Jan-14-2025

We consider the incompressible Navier-Stokes equation in 2 or 3 dimensions. We begin with open $\Omega \subseteq \mathbb{R}^d$. Since the fluid is incompressible, $\partial_t \rho = 0$, so we set $\rho = \rho_0$. The fluid is viscous. The resulting equation is given by:

$$\underbrace{\rho_0 (\partial_t u + (u \cdot \nabla) u)}_{\substack{\text{acceleration,} \\ \text{material derivative}}} = \underbrace{\underbrace{-\nabla p}_{\text{pressure}} + \underbrace{\nu \Delta u}_{\text{viscosity}} + \underbrace{f}_{\text{body force}}}_{\text{Forces}}$$

$$\operatorname{div}(u) = 0$$

with $\quad u_0(x) = u(t,x) \quad$ and $\quad g(x,t) = u(x,t)\big|_{\partial\Omega}$.

Assume $\rho_0 = 1$ by scaling. Book for reference is Chorin & Marsden. For our purposes we take $g \equiv 0$. In practice, $f = 0$, but mathematically it formalizes coupling. For example, with temperature:

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla p + \nu \Delta u + f(T), \ \operatorname{div}(u) = 0 \\ \partial_t T - K \Delta T + (u \cdot \nabla) T = 0 \end{cases}$$

Of course, this is a hard problem. But we can look at simpler problems to try and find our footing. Example, viscous Burgers:

$$\partial_t u + u \partial_x u - \nu \partial_{xx} u = 0.$$

Burgers: $\quad \partial_t u + u \partial_x u = 0, \quad u_0(x) = \varphi(x)$.
To solve this problem, it can be shown that

$u(x,t) = \varphi(x - u(x,t))$. When does a solution exist?
~~We apply the Implicit Function Theorem.~~
~~To be formal, we set $F(u,\varphi) = u - \varphi(x - tu)$. Then, the~~
~~Jacobian is given by:~~

$$\cancel{\partial_u F = 1 + t\varphi'(x - tu)}$$

~~Thus we require $1 + t\varphi'(x - tu) \neq 0$. Note that we already~~
~~have that $F(u,\varphi) = 0$ at $t = 0$, so if the determinant~~
~~is nonzero we have a solution locally.~~ In particular,
if $\varphi' \geq 0$, we have a global solution.

IFT: Let $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be continuously differentiable, with $(a,b) \in \mathbb{R}^n \times \mathbb{R}^m$ such that $f(a,b) = 0$. If the Jacobian matrix of f with respect to $\mathbb{R}^m$:

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_m} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

is invertible at (a,b), then there exists an open set $U \in \mathbb{R}^n$ with $a \in U$ and a unique function $g: U \to \mathbb{R}^m$ such that $g(a) = b$ and

$$f(x, g(x)) = 0 \quad \forall x \in U.$$

In addition, g is continuously differentiable.

Let us apply this theorem to Burgers. We set $F: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ by $F(x,t,y) = y - \varphi(x - ty)$. Set $y = u_0(x)$, and by assumption $F(x, 0, u_0(x)) = u_0(x) - \varphi(x) = 0$. Then, $\partial_y F = 1 + t\varphi'(x - ty)$, so $\partial_y F(x, 0, u_0(x)) = 1 \neq 0$. Thus a solution exists (at least

Locally, if $\varphi' > 0$, global solution.

Let's talk about integrals: Suppose we want to minimize the following:

$$\mathcal{E}(u) := \int_D |\nabla u|^2 - fu\,dx = 0.$$

Let u^* be the minimizer. Then, $\forall \varphi \in C_c^\infty(D)$, $\varepsilon \in \mathbb{R}$
$\mathcal{E}(u^* + \varepsilon\varphi) \geq \mathcal{E}(u^*)$. Hence:

$$\begin{aligned}\mathcal{E}(u^* + \varepsilon\varphi) &= \int_D |\nabla u^* + \varepsilon\nabla\varphi|^2 - fu^* - f\varepsilon\varphi\,dx \\ &= \int_D |\nabla u^*|^2 + 2\varepsilon\nabla u^*\cdot\nabla\varphi + \varepsilon^2|\nabla\varphi|^2 - fu^* - f\varepsilon\varphi\,dx \\ &\geq \int_D |\nabla u^*|^2 - fu^*\,dx = \mathcal{E}(u^*).\end{aligned}$$

Thus, $\int_D 2\varepsilon\nabla u^*\cdot\nabla\varphi + \varepsilon^2|\nabla\varphi|^2 - f\varepsilon\varphi\,dx \geq 0$. Then

$\varepsilon^2|\nabla\varphi|^2 \geq 0$, so, $\forall \varepsilon \in \mathbb{R}$,

$\varepsilon\int_D 2\nabla u^*\cdot\nabla\varphi - f\varphi\,dx \geq 0$. Since this is $\varepsilon \in \mathbb{R}$, we

have again $\int_D 2\nabla u^*\cdot\nabla\varphi - f\varphi\,dx = 0 \quad \forall \varphi \in C_c^\infty(D)$. If we let

$u \in C^2$, then $\int_D (-2\Delta u - f)\varphi\,dx = 0 \quad \forall \varphi \in C_c^\infty(D)$, and so
$-2\Delta u - f = 0$. Can reduce to $u \in H^2$.

And so by minimizing a functional we derived the PDE. We can also go from the PDE to the functional, and see what is being minimalized.

Jan-16-2025

So we want to use integrals to understand what's going on, i.e, notion of weak solutions. Enter distributions: $\Omega \subseteq \mathbb{R}^d$ is our domain, $C_0^\infty(\Omega)$ is clearly a linear space. But we would like a topology to discuss limits. So, we define closed sets via limits.

We start with some notation. We take $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha_j \in \mathbb{N} \cup \{0\}$, and $|\alpha| = \sum_{j=1}^{d} \alpha_j$. Now we define the operator:

$$D^\alpha g := \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}} g.$$

Def: Let $\{\varphi_n\} \subset C_0^\infty(\Omega)$, $\varphi \in C_0^\infty(\Omega)$. We say $\varphi_n \to \varphi$ in distribution if, $\exists K \subset\subset \Omega$, $\forall \varepsilon > 0$, $\forall m \in \mathbb{N}$, $\exists N_0(\varepsilon, m)$ s.t. $\forall |\alpha| \le m$ $\forall n \ge N_0$ $\| D^\alpha \varphi_n - D^\alpha \varphi \|_u < \varepsilon$, $\operatorname{supp}(\varphi) \subset K$. Or, $\exists K \subset\subset \Omega$ and $\forall m \in \mathbb{N}$, $|\alpha| \le m$, $D^\alpha \varphi_n \to D^\alpha \varphi$ uniformly.

With this topology, we say $(C_0^\infty(\Omega), \mathcal{T}) =: \mathcal{D}$. Note that $\mathcal{D}$ is not a Banach space. But we can still talk about a dual.

Def: $\mathcal{D}'$ is the span of continuous (not bounded!) linear functionals on $\mathcal{D}$.

Of course continuity means, if $\varphi_n \to \varphi$ in $\mathcal{D}$, then $\mu \in \mathcal{D}'$ has $\langle \mu, \varphi_n \rangle \to \langle \mu, \varphi \rangle$.

Prop: Let $f \in L^1_{loc}$. For $\varphi \in \mathcal{D}$, define $\langle \mu_f, \varphi \rangle := \int_\Omega f \varphi \, dx$. Then $\mu_f \in \mathcal{D}'$.

Proof: Linearity is clear from properties of integrals. For continuity, suppose $\varphi_n \to \varphi$. So there is compact $K \subset \Omega$ such that, $\forall \alpha$, $D^\alpha \varphi_n \to D^\alpha \varphi$ uniformly on K; in particular $\varphi_n \to \varphi$ uniformly on K. Then:

$$|\langle \mu_f, \varphi_n \rangle - \langle \mu_f, \varphi \rangle| = \left| \int_K f \varphi_n - f \varphi \, dx \right| \le \int_K |\varphi_n - \varphi| |f| \, dx \le \sup_{x \in K} |\varphi_n - \varphi| \int_K |f| \, dx \to 0. \quad \blacksquare$$

Next suppose $f \in C^1 \subseteq L^1_{loc}$, and $g = \frac{\partial}{\partial x_k} f$. Then the induced distribution of f has

$$\langle \mu_g, \varphi \rangle = \int_\Omega f \varphi \, dx = \int_\Omega \frac{\partial}{\partial x_k} g \, \varphi \, dx = (-1) \int_\Omega g \frac{\partial}{\partial x_k} \varphi \, dx.$$

$$= \langle \mu_g, \frac{\partial}{\partial x_k} \varphi \rangle.$$

Def: Let $\mu \in \mathcal{D}'$. Then $\frac{\partial}{\partial x_k} \mu$ is defined by $\langle \frac{\partial}{\partial x_k} \mu, \varphi \rangle := \langle \mu, \frac{\partial}{\partial x_k} \varphi \rangle$.

Prop: For $\mu \in \mathcal{D}'$, $\frac{\partial}{\partial x_k} \mu \in \mathcal{D}'$.

Proof: again, linearity is clear. We show continuity. Suppose $\varphi_n \to \varphi$. Then, since $\{\varphi_n\} \subset C_0^\infty(\Omega)$, $\varphi \in C_c^\infty(\Omega)$, $\{\partial_{x_k}\varphi_n\} \subset C_0^\infty(\Omega)$ and $\partial_{x_k}\varphi \in C_0^\infty(\Omega)$, and $\partial_{x_k}\varphi_n \to \partial_{x_k}\varphi$ in $\mathcal{D}$. Thus,

$$\langle \partial_{x_k}\mu, \varphi_n\rangle := \langle \mu, \partial_{x_k}\varphi_n\rangle \to \langle \mu, \partial_{x_k}\varphi\rangle = \langle \partial_{x_k}\mu, \varphi\rangle. \quad \blacksquare$$

And we can continue to define derivatives for distributions.

A few things to observe: $\forall f \in L^p(\Omega)$ $1 \le p \le \infty$, $\langle \mu_f, \varphi\rangle := \int_\Omega f\varphi\, dx$ defines a distribution, since $\text{supp}(\varphi) \subset K$, $|\int_\Omega f\varphi\, dx| = |\int_K f\varphi\, dx| \le \|\varphi\| |\int f\, dx| \le \|\varphi\| \mu(K)^q \|f\|_{L^p}^p < \infty$. w/ $\frac{1}{p}+\frac{1}{q}=1$, and so well defined. In this respect, we can abuse notation and write $L^p(\Omega) \subset \mathcal{D}'(\Omega)$, and in particular, $L^p(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$. (?)

It is not the case that $\forall \mu \in \mathcal{D}'$, $\exists f \in L^1_{loc}$ so that $\langle \mu, \varphi\rangle = \int f\varphi\, dx$. Consider $\langle \mu, \varphi\rangle = \varphi(0)$, $\Omega \subset \mathbb{R}$, $(0,1) \subset \Omega$. Suppose there was. Let $\varphi \in C_0^\infty(\Omega)$ s.t. $\varphi(0) \neq 0$. Define $\varphi_n(x) := \varphi(nx)$. Then, by compact support, $\varphi_n \to 0$ a.e., with $\varphi_n(0) = \varphi(0)$. Since φ_n are bounded on a set of finite measure, by DCT $\int_\Omega \varphi_n f\, dx \to 0$. But the assumption was $\int_\Omega \varphi_n f\, dx = \varphi_n(0) = \varphi(0) \neq 0$. $\blacksquare$

Sobolev Spaces: So now, for $f \in L^p$, we can take $\mu_f \in \mathcal{D}'$, and so $D^\alpha \mu_f \in \mathcal{D}'$ is well defined. If there is a $g \in L^p$ such that $\langle D^\alpha \mu_f, \varphi\rangle = \int_\Omega g\varphi\, dx$ $\forall \varphi$, $\forall \alpha$, $|\alpha| \le m$, $m \in \mathbb{N}$
is depend on α.

and say $D^\alpha f = g$.

Then we say $f \in W^{m,p}(\Omega)$, and $W^{m,p}(\Omega)$ is called a Sobolev space; it is a Banach Space with norm:

$$\|f\|_{W^{m,p}} := \sum_{0 \le |\alpha| \le m} \|D^\alpha f\| \frac{L^{|\alpha|}}{L^m}, \quad L = \text{length scale.}$$

When $p=2$, these are Hilbert Spaces.

Now to alleviate boundary problems, we consider NS with periodic boundary conditions. That is, we have a fundamental domain $[0,L]^d$, and $f(x+Le_j) = f(x)$ for $j \in \{1:d\}$. But now we need to redefine $\mathcal{D}$. Take $C_p^\infty(\mathbb{R}^n)$ to be C^∞ functions with periodic boundary conditions. Then, $\varphi_n \to \varphi$ if, $\forall K$ compact $D^\alpha \varphi_n \to D^\alpha \varphi$ uniformly. But since they have periodic b.c., this means $D^\alpha \varphi_n \to D^\alpha \varphi$ uniformly on $\mathbb{R}^n$. So we can take that as our definition. $\mathcal{D}_p'$ is the dual.

Jan-23-2025

Preliminaries

Suppose $\frac{1}{p} = \frac{1}{q} - \frac{m}{d} > 0$. Then the following, called the Sobolev Embedding Theorem, holds:

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega).$$

That is, the identity map $i : W^{m,p}(\Omega) \to L^q(\Omega)$ is continuous, or equivalently bounded: $\exists\, C > 0$ s.t. $\|u\|_{L^q} \le C \|u\|_{W^{m,p}(\Omega)}$. If Ω is compact, then then the embedding is compact: bounded sets in $W^{m,p}(\Omega)$ map to pre-compact sets in $L^q(\Omega)$.

Fourier Series: Let $\Omega = \mathbb{T}^d = [0,L]^d$. Let $\varphi \in L^1(\Omega)$.

Define $\hat{\varphi}_k := \int_\Omega \varphi(x) e^{\frac{-2\pi i k\cdot x}{L}} dx$, $k \in \mathbb{Z}^d$, $x \in \mathbb{R}^d$, $x \in \Omega$.
↳ Fourier coefficients.

We can then map φ to its coefficients, $\varphi \to \{\hat{\varphi}_k\}$, and make the association $\varphi(x) \sim \sum_{k \in \mathbb{Z}^d} \hat{\varphi}_k e^{\frac{2\pi i k\cdot x}{L}} dx$

Prop: Suppose $\varphi \in L^2(\Omega) \subset L^1(\Omega)$. Let $S_N(\varphi)(x) = \sum_{|k| \le N} \hat{\varphi}_k e^{\frac{2\pi i k\cdot x}{L}}$. Then, $\|\varphi - S_N\|_{L^2} \to 0$ as $N \to \infty$. So in L^2,

$$\varphi(x) = \sum_{k \in \mathbb{Z}^d} \hat{\varphi}_k e^{\frac{2\pi i k\cdot x}{L}}.$$

If $\varphi \in \mathbb{R}$, then $\overline{\hat{\varphi}_k} = \overline{\int_\Omega \varphi(x) e^{\frac{-2\pi i k\cdot x}{L}} dx} = \int_\Omega \varphi(x) e^{\frac{-2\pi i(-k)\cdot x}{L}} dx = \hat{\varphi}_{-k}$. Thus, $\hat{\varphi}_{-k} + \hat{\varphi}_k$ is real valued, and so $S_N(\varphi)(x) \in \mathbb{R}$, and so $\sum_{k \in \mathbb{Z}^d} \hat{\varphi}_k e^{\frac{2\pi i k\cdot x}{L}} \in \mathbb{R}$. (Good reality check.)

Inner products: $f,g \in \mathbb{C}$, $(f,g) := \int_\Omega f\bar{g}\,dx$. Then, since the series converges, we obtain

$$(f,g) = \int_\Omega \Big(\sum_{k\in\mathbb{Z}^d} \hat{f}_k e^{\frac{2\pi i x\cdot k}{L}}\Big)\Big(\overline{\sum_{m\in\mathbb{Z}^d} \hat{g}_m e^{\frac{2\pi i x\cdot m}{L}}}\Big)dx$$

$$= \sum_k \sum_m \int_\Omega \hat{f}_k \bar{\hat{g}}_m \underbrace{e^{\frac{2\pi i x\cdot k}{L}} \cdot e^{\frac{2\pi i x\cdot(-m)}{L}}}_{\delta_{k,m}\text{, since } e^{\frac{2\pi i x\cdot k}{L}} \text{ is on basis}} dx$$

$$= \sum_k \hat{f}_k \bar{\hat{g}}_m$$

Thus, $(f,f) = \sum_k |\hat{f}_k|^2$. (Parseval's Identity)

In particular, $|\hat{f}_k| \longrightarrow 0$ as $k \to \infty$, and so

$$\left.\begin{array}{l} \int f(x)\cos\left(\frac{2\pi k\cdot x}{L}\right)dx \to 0 \\ \int f(x)\sin\left(\frac{2\pi k\cdot x}{L}\right)dx \to 0 \end{array}\right\} \text{Riemann Lebesgue Lemma}$$

Now back to Sobolev: Suppose $\varphi \in L^2_p(\Omega)$, with $\nabla\varphi \in L^2_p(\Omega)$. Then,

$$\widehat{\partial_x \varphi}_k = \int_D \partial_x \varphi\, e^{\frac{-2\pi i k\cdot x}{L}}dx = -\int_D \varphi\, \partial_x e^{\frac{-2\pi i k\cdot x}{L}}dx = \frac{2\pi i k_1}{L}\int_D \varphi e^{\frac{-2\pi i k\cdot x}{L}}dx$$

$= \frac{2\pi i k_1}{L}\hat{\varphi}_k$. Similarly, $\widehat{\partial_y \varphi}_k = \frac{2\pi i k_2}{L}\hat{\varphi}_k$. Hence, we can say

$$\widehat{(\nabla\varphi)}_k = \frac{2\pi i}{L} k\, \hat{\varphi}_k.$$

Then by Parseval's: $\|\nabla\varphi\|^2_{L^2} = \sum_k \left(\frac{2\pi}{L}\right)^2 |k|^2 |\hat{\varphi}_k|^2$, or

$$\|\nabla\varphi\|_{L^2} = \left(\frac{2\pi}{L}\right)\Big(\sum_k |k|^2 |\hat{\varphi}_k|^2\Big)^{1/2}.$$

Jan-23-2025

Prelims:

$W^{m,p}(\Omega) \hookrightarrow L^{q}(\Omega)$, $\frac{1}{q} = \frac{1}{p} - \frac{m}{d} > 0$,

Sobolev Embedding.

→ If $\Omega = K$, then inclusion is compact.

Fourier Series: $\Omega = \mathbb{T}^d = [0, L]^d$ exponent needs to be dimensionless.

Let $\varphi \in L^1_{per}(\Omega)$, $\underbrace{\hat{\varphi}_k := \int_\Omega \frac{\varphi(x) e^{\frac{-2\pi i x \cdot k}{L}}}{L^d} dx}_{\text{Fourier coefficients}}$, $k \in \mathbb{Z}^d$, $x \in \mathbb{R}^d$, $x \in \Omega$.

$\varphi \to \{\hat{\varphi}_k\}$

$\varphi(x) \sim \sum_{k \in \mathbb{Z}^d} \hat{\varphi}_k e^{\frac{2\pi i k \cdot x}{L}}$ (association)

Prop: Suppose $\varphi \in L^2_p(\Omega) \subset L^1_p(\Omega)$, $\varphi \to \{\hat{\varphi}_k\}$,

$S_N(\varphi)(x) := \sum_{|k| \le N} \hat{\varphi}_k e^{\frac{2\pi i k \cdot x}{L}} \in \mathbb{C}$. Then:

$\|\varphi - S_N(\varphi)\|_{L^2_p(\Omega)} \to 0$ $N \to \infty$, i.e. $\varphi(x) = \sum_{k \in \mathbb{Z}_d} \hat{\varphi}_k e^{\frac{2\pi i k x}{L}}$ (in L^2)

And if $\varphi \in \mathbb{R}$, then $\overline{\hat{\varphi}_k} = \hat{\varphi}_{-k}$ (Reality condition).

· inner product in L^2, $\int_\Omega f \bar{g}\, dx = (f, g) \underset{\text{expansion}}{=} \int \sum_{k \in \mathbb{Z}^d} \hat{f}_k e^{\frac{2\pi i x \cdot k}{L}} \cdot \sum_{m \in \mathbb{Z}^d} \bar{\hat{g}}_m e^{\frac{-2\pi i x \cdot m}{L}} dx$

$= \sum_{m,k \in \mathbb{Z}^d} \int_\Omega \hat{f}_k \bar{\hat{g}}_m \underbrace{e^{\frac{2\pi i (k-m) \cdot x}{L}}}_{k \ne m \Rightarrow 0} dx = \sum_{n \in \mathbb{Z}^d} \hat{f}_n \bar{\hat{g}}_n \Rightarrow \|\varphi\|^2_{L^2} = \sum_{k \in \mathbb{Z}^d} |\hat{f}_k|^2$

$\Rightarrow \hat{\varphi}_k \to 0$ as $k \to \infty$, so $\int \varphi(x) e^{-\frac{2\pi i x \cdot k}{L}} dx \xrightarrow[k \to \infty]{} 0 \Rightarrow$

$\left.\begin{array}{l} \int \varphi(x) \cos\left(\frac{2\pi k \cdot x}{L}\right) dx \to 0 \\ \int \varphi(x) \sin\left(\frac{2\pi k \cdot x}{L}\right) dx \to 0 \end{array}\right\}$ Riemann Lebesgue.

Derivative units.

and so we can define $\|\varphi\|^2_{H^1} := \|\varphi\|^2_{L^2} + \|\nabla\varphi\|^2 =$

$$\sum_k \frac{|\hat{\varphi}_k|^2}{L^2} + \frac{(2\pi)^2}{L^2}|k|^2|\hat{\varphi}_k|^2 \simeq \frac{1}{L^2}\sum_k (1+|k|^2)|\hat{\varphi}_k|^2.$$

↳ So can use this as norm on H^1.

↳ and! $|\hat{\varphi}_k|^2$ decays faster than any poly of the degree!

We can continue this way by:

$$\|\varphi\|^2_{H^m_p(\Omega)} := \underbrace{\frac{1}{L^{2m}}\left(\sum_{k\in\mathbb{Z}} (1+|k|^2)^m|\hat{\varphi}_k|^2\right)}_{}$$

comes from the expansion and some equivalences of polynomials.

and so we define $H^s_p(\Omega) := \{\varphi \in L^2 : \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^s|\hat{\varphi}_k|^2 < \infty\}$, for any $s>0$, with

$$\|\varphi\|_{H^s_p(\Omega)} := \left(\sum_k (1+|k|^2)^s|\hat{\varphi}_k|^2\right)^{1/2}\frac{1}{L^s}.$$

<u>Fourier Transform</u>: on $\mathbb{R}^d$ $\mathcal{F}(f)(\xi) := \hat{f}(\xi) =$

⟶ ξ has 1/L units!

$\int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi}dx$, but this exists for $f\in L^1$. When $f\in L^2(\mathbb{R}^d)$, we first look at $C^\infty_c(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$:

$$\mathcal{F}(\varphi)(\xi) = \int_{\mathbb{R}^d}\varphi(x)e^{-ix\cdot\xi}dx.$$

Now, for smooth functions, it can be shown $\int_{\mathbb{R}^d} |\hat{\varphi}(\xi)|^2 d\xi \simeq \int_{\mathbb{R}^d} |\varphi(x)|^2 dx$. So for $g \in L^2(\mathbb{R}^d)$, choose $\{\varphi_k\} \subset C_c^\infty(\mathbb{R}^d)$ s.t. $\varphi_k \to g$ in L^2. Then,

$$\| \mathcal{F}(\varphi_k) - \mathcal{F}(\varphi_m) \| \simeq \underbrace{\| \varphi_k - \varphi_m \|}_{\text{Cauchy}}$$

so $\{\mathcal{F}(\varphi_k)\}$ converges in L^2. So we may define $\mathcal{F}(g)$ as the limit.

Compactness Theorem: Suppose $0 \leq s_1 < s_2$. Then $H_p^{s_2}(\Omega) \hookrightarrow H_p^{s_1}(\Omega)$ is continuous and compact.

Interpolation Theorem: Suppose $0 \leq s_1 < s < s_2$. $s = \theta_1 s_1 + (1-\theta) s_2$. Then, $\|\varphi\|_{H^s} \leq \|\varphi\|_{H^{s_1}}^{\theta} \|\varphi\|_{H^{s_2}}^{1-\theta}$.

Proof: $\|\varphi\|_{H^s}^2 = \int (1+|\xi|^2)^s |\hat{\varphi}(\xi)|^2 d\xi$

$$= \int (1+|\xi|^2)^{\theta s_1} |\hat{\varphi}(\xi)|^{2\theta} (1+|\xi|^2)^{1-\theta} |\hat{\varphi}(\xi)|^{2(1-\theta)} d\xi$$

Holder $\leq \|\varphi\|_{H^{s_1}}^{2\theta} \|\varphi\|_{H^{s_2}}^{2(1-\theta)}$, $p = \frac{1}{\theta}$, $q = \frac{1}{1-\theta}$ for Holder. ■

Corollary: $\underbrace{\|u\|_{L^q} \leq c \|u\|_{H^{s_1}}^{\theta}}_{\text{Sobolev Embedding}} \underbrace{\|u\|_{H^{s_2}}^{1-\theta}}_{\text{Interpolation}}$ for $\frac{1}{q} = \frac{1}{2} - \frac{s}{d}$, $s_1 < s < s_2$

Jan-28-2025

We have the interpolation for $q < \infty$ - does it hold for $q = \infty$. Yes! Called Agmon's inequality.

<u>Prop</u>: Let $0 \le s_1 < s < s_2$, with $s = s^* = \frac{d}{2}$ (so SET doesn't help). Then $\|u\|_{L^\infty} \le c\|u\|_{H^{s_1}}^{\theta}\|u\|_{H^{s_2}}^{1-\theta}$.

<u>Proof</u>: Using inverse Fourier, we have $u(x) = \int_{\mathbb{R}^d} \hat{u}(\xi) e^{i x \cdot \xi} d\xi$

Then,
$$|u(x)| \le \int_{\mathbb{R}} |\hat{u}(\xi)| d\xi = \int_{|\xi| \le R} |\hat{u}(\xi)| d\xi + \int_{|\xi| > R} |\hat{u}(\xi)| d\xi$$

$$= \int_{|\xi| \le R} |\hat{u}(\xi)| \frac{(1+|\xi|^2)^{s_1/2}}{(1+|\xi|^2)^{s_1/2}} d\xi + \int_{|\xi| > R} |\hat{u}(\xi)| \frac{(1+|\xi|^2)^{s_2/2}}{(1+|\xi|^2)^{s_2/2}} d\xi$$

$$\le \left(\int_{|\xi| \le R} |\hat{u}(\xi)|^2 (1+|\xi|^2)^{s_1} d\xi\right)^{1/2} \left(\int_{|\xi| \le R} \frac{1}{(1+|\xi|^2)^{s_1}} d\xi\right)^{1/2} +$$

$$\left(\int_{|\xi| > R} |\hat{u}(\xi)|^2 (1+|\xi|^2)^{s_2} d\xi\right)^{1/2} \left(\int_{|\xi| > R} \frac{1}{(1+|\xi|^2)^{s_2}} d\xi\right)^{1/2}$$

$$\le \|u\|_{H^{s_1}} \underbrace{\left(\int_{|\xi| \le R} \frac{1}{(1+|\xi|^2)^{s_1}} d\xi\right)}_{I_1(R)} + \|u\|_{H^{s_2}} \underbrace{\left(\int_{|\xi| > R} \frac{1}{(1+|\xi|^2)^{s_2}} d\xi\right)}_{I_2(R)}$$

$$\Rightarrow |u(x)| \le \|u\|_{H^{s_1}} I_1(R) + \|u\|_{H^{s_2}} I_2(R).$$

Now, $I_1(R)^2 = \int_{|\xi| \le R} \frac{1}{(1+|\xi|^2)^{s_1}} d\xi = \int_{S^{d-1}} \int_0^R \frac{\rho^{d-1}}{(1+\rho^2)^{s_1}} d\rho\, d\omega$

$$= A(S^{d-1}) \int_0^R \frac{\rho^{d-1}}{(1+\rho^2)^{s_1}} d\rho \le A(S^{d-1}) \int_0^R \rho^{d-1-2s_1} d\rho$$

$$= A(S^{d-1}) \frac{R^{d-2s_1}}{d-2s_1}$$

$$I_2(R)^2 = \int_{|\xi| \ge R} \frac{1}{(1+|\xi|^2)^{s_2}} d\xi = \int_{S^{d-1}} \int_R^\infty \frac{\rho^{d-1}}{(1+\rho^2)^{s_2}} d\rho\, d\omega$$

$$\le A(S^{d-1}) \int_R^\infty \rho^{d-1-2s_2} d\rho = A(S^{d-1}) \frac{R^{d-2s_2}}{2s_2-d}$$

$\hookrightarrow d-2s_2 < 0$ since $2s_2 > d$.

$$\Rightarrow \|u\|_{L^\infty(\Omega)} \le \|u\|_{H^{s_1}} \cdot A(S^{d-1})^{1/2} \underbrace{\left(\frac{R^{d-2s_1}}{d-2s_1}\right)^{1/2}}_{\substack{2s_1 < d \\ \to \infty \text{ as } R \to \infty \\ \to 0 \text{ as } R \to 0}} + \|u\|_{H^{s_2}} \underbrace{\left(\frac{R^{d-2s_2}}{2s_2-d}\right)^{1/2}}_{\substack{2s_2 > d \\ \to 0 \text{ as } R \to \infty \\ \to \infty \text{ as } R \to 0}} A(S^{d-1})^{1/2}$$

$\Rightarrow$ Find optimal R.

Achieve $\|u\|_{L^\infty(\Omega)} \le C\left(\|u\|_{H^{s_1}} + \|u\|_{H^{s_2}}\right)$

$\hookrightarrow$ Unsure how to go further...

Using this result, obtain:

2D: $s^* = 1$ $\quad \|u\|_{L^\infty} \le C\|u\|_{L^2}^{1/2}\|u\|_{H^2}^{1/2}$

3D: $s^* = 3/2$ $\quad \|u\|_{L^\infty} \le \left\{ \begin{array}{l} C\|u\|_{H^1}^{1/2}\|u\|_{H^2}^{1/2} \\ C\|u\|_{L^2}^{1/4}\|u\|_{H^2}^{3/4} \end{array} \right\} \rightarrow$ These are the only exponents that work!

Interpolation:

2D: $q=4 \rightarrow \frac{1}{q} = \frac{1}{2} - \frac{s}{2} \Rightarrow s = \frac{1}{2}$, $\|u\|_{L^4} \le C\|u\|_{H^{1/2}}$
$\le C\|u\|_{L^2}^{1/2}\|u\|_{H^1}^{1/2}$

3D: $q=4 \rightarrow \frac{1}{q} = \frac{1}{2} - \frac{s}{3} \Rightarrow s = \frac{3}{4}$, $\|u\|_{L^4} \le C\|u\|_{H^{3/4}}$
$\le C\|u\|_{L^2}^{1/4}\|u\|_{H^1}^{3/4}$

$q=3 \rightarrow \frac{1}{q} = \frac{1}{2} - \frac{s}{3} \Rightarrow s = \frac{1}{2}$ $\quad \|u\|_{L^3} \le C\|u\|_{H^{1/2}} \le C\|u\|_{L^2}^{1/2}\|u\|_{H^1}^{1/2}$

$q=6 \Rightarrow \|u\|_{L^6} \le C\|u\|_{H^1}$

$\rightarrow$ The whole problem with NS in 3D! These exponents are the best you can get; this is the best we can get with this approach.

Back to distributions... $\Omega = \mathbb{T}^d = (\mathbb{R}/_{\mathbb{Z}})^d$. Recall $\mathcal{T}(\Omega)$ dense in $\mathcal{D}(\Omega)$.

For $f \in L^2$, we have $\hat{f}_k = \frac{1}{L^d}\int_\Omega f(x) e^{\frac{-2\pi i k \cdot x}{L}} dx$, and

$(f,g)_{L^2} = \sum_k \hat{f}_k \bar{\hat{g}}_k$.

Def: Let $\mu \in D'(\Omega)$. Define $\hat{\mu}_k := \langle \mu, e^{\frac{-2\pi i k \cdot x}{L}} \rangle \frac{1}{L^d}$.

Claim: $\langle \mu, \varphi \rangle = \sum_k \mu_k \bar{\hat{\varphi}}_k$.

Proof: $S_N(\varphi)(x) = \sum_{|k| \le N} \hat{\varphi}_k e^{\frac{2\pi i k \cdot x}{L}}$, and $S_N(\varphi) \to \varphi$ in $D(\Omega)$!

Since trig polys are dense in $D(\Omega)$. Then,

$$\langle \mu, S_N(\varphi) \rangle \longrightarrow \langle \mu, \varphi \rangle, \text{ but } \langle \mu, S_N(\varphi) \rangle = \sum_{|k| \le N} \langle \mu, \hat{\varphi}_k e^{\frac{2\pi i k \cdot x}{L}} \rangle$$

$= \sum_{|k| \le N} \bar{\hat{\varphi}}_k \langle \mu, e^{\frac{2\pi i k \cdot x}{L}} \rangle = \sum_{|k| \le N} \hat{\mu}_k \bar{\hat{\varphi}}_k$, so $\sum_k \mu_k \bar{\hat{\varphi}}_k = \langle \mu, \varphi \rangle$. ■

a few more definitions:

$H^1_p(\Omega)$, $(H^1_p)' =$ dual of H^1. "$(H^1(\Omega))' \subset D'(\Omega)$"

Lemma: Let $\mu \in (H^1_p)'$. $\exists \tilde{\mu} \in D'(\Omega)$ s.t.
$\langle \tilde{\mu}, \varphi \rangle = \langle \mu, \varphi \rangle \; \forall \varphi \in D(\Omega)$.

Proof: Let $\varphi \in D(\Omega)$, so $\varphi \in H^1_p(\Omega)$. So, we define
$\langle \tilde{\mu}, \varphi \rangle_{D'} := \langle \mu, \varphi \rangle_{H^1}$. Linearity is clear. Let
$\varphi_n \to \varphi$ in $D(\Omega)$. Then $\varphi_n \to \varphi$ in H^1. So,
$\langle \tilde{\mu}, \varphi_n \rangle_{D'} = \langle \mu, \varphi_n \rangle_{H^1} \longrightarrow \langle \mu, \varphi \rangle_{H} = \langle \tilde{\mu}, \varphi \rangle_{D'}$ ■

Ex: Let $b_k \in \mathbb{R}$. Does $\exists\, \mu \in \mathcal{D}'$ s.t. $\hat{\mu}_k = b_k$? On what condition. Yes, if $\forall \varphi \in \mathcal{D}(\Omega)$, $\sum_{k \in \mathbb{Z}^d} |b_k||\hat{\varphi}_k| < \infty$, and then

↳ Prove this.

$\langle \mu, \varphi \rangle = \sum b_k \hat{\varphi}_k$ ↳ iff?

Exercise: Let $\{b_k\} \in \mathbb{R}$. Does $\exists\, \mu \in \mathcal{D}'(\Omega)$ s.t. $\hat{\mu}_k = b_k$? Yes, if $\forall \varphi \in \mathcal{D}(\Omega)$, $\sum_k |b_k| |\hat{\varphi}_k| < \infty$, and then $\langle \mu, \varphi \rangle = \sum_k b_k \hat{\varphi}_k$.

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Back to H^1:

$$\|u\|_{(H^1)'} = \sup_{\|\varphi\|=1} |\langle \mu, \varphi \rangle|_{(H^1)' \times H^1} = \sup_{\|\varphi\|=1} \left| \sum_k \hat{\mu}_k \bar{\hat{\varphi}}_k \right| L^d \leq$$

$$\sup_{\|\varphi\|_{H^1}=1} \sum_k |\hat{\mu}_k| |\bar{\hat{\varphi}}_k| \leq \sup_{\|\varphi\|=1} \sum_k |\hat{\mu}_k| |\bar{\hat{\varphi}}_k| \frac{(1+|\hat{\varphi}_k|^2)^{1/2}}{(1+|\hat{\varphi}_k|^2)^{1/2}}$$

$$\leq \sup_{\|\varphi\|=1} \left(\sum_k \frac{|\mu_k|^2}{(1+|\hat{\varphi}_k|^2)} \right)^{1/2} \underbrace{\left(\sum_k |\hat{\varphi}_k|^2 (1+|\hat{\varphi}_k|^2) \right)^{1/2}}_{\|\varphi\|_{H^1}} L^{1/2} \cdot L^{1/2}$$

$$= \sum_k |\mu_k|^2 (1+|\hat{\varphi}_k|^2)^{-1} L^{1/2}$$ ~ Can show equality!

$$\Rightarrow \|u\|_{(H^1)'} = \sum_k |\mu_k|^2 (1+|\hat{\varphi}_k|^2)^{-1} L^{1/2}.$$

Exercise: Let $\{b_k\} \in \mathbb{C}$, $\bar{b}_k = b_{-k}$. $\exists!\, \mu \in (H^1)'$ s.t. $\hat{\mu}_k = b_k$ iff $\sum_k \frac{|b_k|^2}{1+|k|^2} < \infty$.

Now back to NS: $\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla p = f + \nu \Delta u \\ \operatorname{div}(u) = 0, \ u\text{-periodic.} \end{cases}$

Define $\bar{u} = \frac{1}{|\Omega|} \int_\Omega u(x)\, dx$.

Componentwise: $\partial_t u_k + \sum_{\ell=1}^{d} u_\ell \partial_\ell u_k + \partial_\ell p = f_\ell + \nu \sum_\ell \partial_\ell^2 u_k$

$\overbrace{}^{\text{div}(u)=0}$

Now $\sum_\ell v_\ell \partial_\ell u_K = \sum_{K\ell} \partial_\ell (v_\ell u_K) - \sum_{K\ell} \partial_\ell (v_\ell) u_K$. Thus,

$\frac{d}{dt}\bar{u} = \bar{f}$, so w/ $\bar{f}=0$, $\bar{u}(t) = \bar{u}(0)$, so take $\bar{u}(0)=0$.

<u>Poincare Inequality:</u> For $\varphi \in H^1(\Omega)$

$$\|\varphi - \bar{\varphi}\|_{L^2} \leq C_p(\Omega) \|\nabla \varphi\|_{L^2}.$$

So when $\bar{\varphi}=0$, $\|\varphi\|_{L^2} \leq C_p(\Omega)\|\nabla\varphi\|_{L^2}$.

Define $\dot{X} := \{\varphi \in X : \int \varphi \, dx = 0\}$.

<u>Hemholtz Decomposition:</u> $\text{div}(u)=0$, $p \in \dot{L}^2(\Omega)$

Formally $\int_\Omega u(x)\cdot\nabla p \, dx = -\int_\Omega p \, \text{div}(u) \, dx = 0$. So, $u \perp \nabla p$. So maybe find a decomposition with div-free and 0-average.

$(H^1(\Omega))' \subseteq \mathcal{D}'(\Omega)$.

<u>Theorem:</u> Let $v \in (H^1)'$. $\exists! \, u \in (H^1(\Omega))'$, $p \in \dot{L}^2(\Omega)$ s.t. $v = u + \nabla p$ and $\text{div}(u) = 0$. In distribution!

First a few things, $\langle \mu, 1 \rangle = \langle \mu, e^0 \rangle = \langle \mu, e^{\frac{2\pi i 0 \cdot x}{L}} \rangle = \hat{\mu}_0$.
Then, for $p \in \dot{L}^2(\Omega)$, $\langle p, 1 \rangle = 0$, so $\hat{p}_0 = 0$. and,
$\text{div}(u)=0 \Rightarrow \widehat{(\text{div}(u))} = iK\cdot\hat{u}_k = 0$. So, $\hat{\mu}_K \perp K \Leftrightarrow \text{div}(u)=0$.

<u>Proof:</u> We want $\text{div}(u)=0$, so we want the following

$$\hat{v}_k = \hat{u}_k + \widehat{\nabla p} = \hat{u}_k + \left(\frac{2\pi i K}{L}\right)\hat{p}_K \Rightarrow \hat{p}_0 = 0 \text{ is already set.}$$

$$iK\cdot\hat{v}_k = -\frac{2\pi}{L}|K|^2 \hat{p}_K \Rightarrow \hat{p}_K = \frac{iK}{|K|^2}\left(\frac{L}{2\pi}\right)\cdot\hat{v}_k.$$

$\dot{L}^2_p(\Omega)^d$, $(\dot{H}^2_p(\Omega))^d$

Hemholtz Decomposition:

NS: ∇p, $\mathrm{div}(u)=0$, formally,

$$\int_\Omega u(x)\cdot\nabla p\,dx = \sum_{j=1}^{d}\int_\Omega u_j \frac{\partial}{\partial x_j} p\,dx$$

$$= -\int p \sum_{j=1}^{d} \frac{\partial}{\partial x_j} u\,dx = -\int p\,\mathrm{div}(u)\,dx = 0.$$

$\Rightarrow u \perp \nabla p.$

Theorem: Let $v \in (H^1)'$, $\exists!\, u \in (H^1(\Omega))'$, $p \in \dot{L}^2(\Omega)$ s.t. $\int p\,dx = 0$

$v = u + \nabla p$ and $\mathrm{div}(u) = 0$.

$\hookrightarrow \mu \in \mathcal{D}'(\Omega)$, $\langle \mu, 1\rangle = \hat{\mu}_0$

Proof: $\hat{V}_k = \hat{u}_k + \left(\frac{2\pi i K}{L}\right)\hat{p}_k$; $\mathrm{div}(u)=0 \Leftrightarrow \widehat{(\nabla\cdot u)}_k = \underbrace{iK\cdot\hat{u}_k}_{K\perp\hat{u}_k} = 0.$

$\Rightarrow \quad iK\cdot\hat{V}_k = 0 - \frac{2\pi|K|^2}{L}\hat{p}_k = -\frac{2\pi K^2}{L}\hat{p}_k.$

$\Rightarrow \quad \hat{p}_0 = 0$ b.c. $p \in \dot{L}(\Omega)$. Then $\hat{p}_k = \frac{iK}{|K|^2}\left(\frac{L}{2\pi}\right)\cdot\hat{v}_k$

$\Rightarrow \quad \hat{u}_k = \hat{v}_k + \left(\frac{2\pi i K}{L}\right)\left(\frac{L}{2\pi}\right)\left(\frac{iK\cdot\hat{v}_k}{|K|^2}\right)$

$\Rightarrow \quad \hat{u}_k = \hat{v}_k - \frac{K(K\cdot\hat{v}_k)}{|K|^2} \rightarrow$ determines $u \in (H^1)'$.

do $\sum_K |p_K|^2 < \infty$? $\sum_K \frac{1}{K^2}|\hat{v}_K|^2 \Rightarrow p \in L^2$. ∎

Thus, $\hat{u}_k = \hat{v}_k - \frac{k(k\cdot\hat{v}_k)}{|k|^2}$, so identifies $u \in (H^1)'$.

Now, $\sum_k |\hat{p}_k|^2 = \sum_k \frac{1}{|k|^2}|\hat{v}_k|^2$; since $v \in (H^1)'$, this is finite, so $p \in \dot{L}^2(\Omega)$. ∎

So, for $v \in (H^1)'$ we can find a $u \in (H^1)'$ with $div(u) = 0$ (in distribution) and a $p \in (\dot{L}^2_p)^d$ such that $(u, \nabla p) = 0$ and $v = u + \nabla p$. Hence

$$(H^1)' = \underbrace{(H^1)'_\sigma}_{\text{div-free}} \oplus \underbrace{(H^1)'_g}_{\text{grad}}.$$

Similarly: If $v \in (\dot{L}^2_p(\Omega))^d$, $\exists !\ u \in \dot{L}^2_p(\Omega)$, $p \in \dot{H}^1_p$ s.t. $v = u + \nabla p$. i.e.

$$\dot{L}^2_p = (L_p)_\sigma \oplus (\dot{L}^2_p)_g$$

More formally, define $H := \{u \in \dot{L}^2_p(\Omega) : div(u) = 0\}$

$H^\perp := \{\nabla p \in (\dot{L}^2_p(\Omega))^d : p \in \dot{H}^1_p(\Omega)\}$.

Then, $(\dot{L}^2_p)^d = H \oplus H^\perp$,

$V := \{u \in (\dot{H}^1_p)^d : div(u) = 0\}$

$V^\perp := \{\nabla p : p \in \dot{H}^2_p\}$.

Now denote $\dot{T}(\Omega)$, $\mathcal{V} := \{v \in \dot{T}(\Omega) : div(v) = 0\}$.

Claim:
- H is a closed subspace of $\dot{L}^2$
- V is closed subspace of $\dot{H}_1$
- $H = \overline{\mathcal{V}}$ in $\dot{L}^2$

• $V = \overline{\mathcal{V}}$ in $\dot{H}^1$.

H-inner product $(\varphi, \psi) = \int \varphi \psi \, dx$, $|\varphi|^2 = (\varphi, \varphi)$

V-inner product: $((\varphi, \psi)) = \int \nabla \varphi \cdot \nabla \psi \, dx$,

$\|\varphi\| = ((\varphi, \varphi))$.

Jan-30-2025 (pt. 2)

Shown: Helmholtz Decomposition

$$(H_p^1(\Omega))' = (H_p^1(\Omega))'_\sigma \oplus (H^1(\Omega))'_g$$

$$v = \underset{\operatorname{div}(u)=0}{u} + \underset{\int p\,dx = 0}{\nabla p}$$

$$(\dot{H}_p^1(\Omega))' =: H_p^{-1} = (H^{-1})_\sigma \oplus (H^{-1})_g$$

$$\dot{L}_p^2 = H \oplus H^\perp$$

$$v = u + \nabla p$$

$$\dot{H}^1 = V \oplus V^\perp$$

Def: $\varphi \in \dot{L}_p^2$, $\varphi = \varphi_\sigma + \varphi_g$, $P_\sigma : L_p \to H$, ortho projection.

$$P_\sigma(\varphi) = \varphi_\sigma, \quad (I - P_\sigma)\varphi = \nabla\pi.$$

$\mathcal{V} = \{\varphi \in \dot{\mathcal{T}}(\Omega) : \nabla \cdot \varphi = 0\} \to$ Test functions,

$\mathcal{V} \subset \dot{D}(\Omega)$.

De Rham Theorem: $\vec{l} \in (\dot{D}'(\Omega))^d$ Suppose $\langle \vec{l}, \varphi \rangle = 0 \ \forall \varphi \in \mathcal{V}$. Then, $\exists!\, p \in \dot{D}'(\Omega)$ s.t. $\vec{l} = \nabla p$.

Proof: Let $\varphi = \vec{a}_K e^{\frac{-2\pi i K \cdot x}{L}} \in \mathcal{V}$, requires $\vec{a}_K \cdot iK = 0 \Leftrightarrow K \cdot \vec{a}_K = 0$. ($K$ fixed, $a_K \perp K$.)

$$\langle l, \varphi \rangle = 0 \Rightarrow \langle l, \vec{a}_K e^{\frac{-2\pi i K \cdot x}{L}} \rangle \overset{\text{component}}{=} \hat{\vec{l}}_K \cdot \vec{a}_K = 0. \text{ So,}$$

$$\Leftrightarrow \hat{\vec{l}}_K = \hat{p}_K K, \quad \hat{\vec{l}}_K \cdot K = \hat{p}_K |K|^2 \Rightarrow \hat{p}_K = \frac{\hat{l}_K \cdot K}{|K|^2} \to b_K \in \mathbb{Z}!$$

$\rho \sim \sum_{K\neq 0} \hat{\rho}_K e^{\frac{2\pi i K\cdot x}{L}}$. Claim $\rho \in \mathring{D}'(\Omega)$.

Let $\varphi \in \mathring{D}(\Omega)$, $\langle \rho, \varphi \rangle = \sum \hat{\rho}_K \bar{\hat{\varphi}}_K L^d \to$ Should hold if $\rho \in \mathring{D}'(\Omega)$.

$= \sum_{K\neq 0} \frac{(\hat{\ell}_K \cdot K)\bar{\hat{\varphi}}_K}{|K|^2} \to$ want this convergent.

$= \sum_{K\neq 0} \hat{\ell}_K \cdot \bar{\hat{\Psi}}_K$, $= \langle \vec{\ell}, \Psi \rangle$. want $\hat{\Psi}_K$ to be FC of a test function.

$\hat{\Psi}_K = \frac{\hat{\varphi}_K \cdot K}{|K|^2}$, $\varphi \in C^\infty$, $\Psi = \sum_{K\neq 0} \hat{\Psi}_K e^{\frac{2\pi i K\cdot x}{L}}$ so is test function. $\mathring{D}(\Omega)$

$\Rightarrow$ Series converges! $\varphi_n \to \varphi$

Ex: $\Psi_n \to \Psi$ in $\mathring{D}(\Omega) \Rightarrow \rho \in \mathring{D}'(\Omega)$.

Problem: Given $\{b_K\} \subseteq \mathbb{C}$ $b_K = \bar{b}_K$. does there $\mu \in \mathring{D}'(\Omega)$ s.t. $\hat{\mu}_K = b_K$. In gen, no. If so,

$\langle \mu, \varphi \rangle = \sum_K \hat{\mu}_K \bar{\hat{\varphi}}_K = \sum_K b_K \bar{\hat{\varphi}}_K$, $\varphi \in C_p^\infty$.

Take $\varphi(x) = \sum_{K\neq 0} e^{-|K|^2} e^{\frac{2\pi i kx}{L}} \to$ Complex Analytic, C_p^∞.

Let $b_K = e^{|K|^2}$ but $\sum e^{|K|^2 - |K|} < \infty? \to$ No! So no luck. Check.

when low Reynolds #, drop non-linearity!

Steady State NS: $\begin{cases} -\nu\Delta u + (u\cdot\nabla)u + \nabla p = f \\ \mathrm{div}(u) = 0 \end{cases}$

$\downarrow$

Stokes: $\begin{cases} -\nu\Delta u + \nabla p = f, \ \nu > 0, \ f \text{ given.} \\ \mathrm{div}(u) = 0 \end{cases}$

Find (u,p). Case $f \in H^{-1} = (H^1)'$, use Helmholtz.

$f = f_\sigma + \nabla g$, $\mathrm{div}(f_\sigma) = 0$, $f_\sigma \in H^{-1}$, $g \in L^2$, or

$f_\sigma = \mathbb{P}(f)$, $g = (I - \mathbb{P})(f)$. $\Rightarrow$

$\begin{cases} -\nu\Delta u + \nabla p = f_\sigma + \nabla g \\ \mathrm{div}(u) = 0 \end{cases}$; but $\nabla\cdot(\Delta u) = \Delta(\nabla\cdot u) = 0$,

so LHS is decomposed: so

$-\nu\Delta u = f_\sigma$, $\nabla p = \nabla g \rightarrow \nabla(p - g) = 0$

$\Updownarrow$ b.c. 0-avg.

$\boxed{p = g}$

Use $\mathcal{F} \Rightarrow \begin{cases} \nu|K|^2\hat{u}_K = (\hat{f_\sigma})_K \ ✓ \\ iK\cdot\hat{u}_K = 0 \ ✓ \rightarrow \text{Solve this system.} \end{cases}$

$\Rightarrow \hat{u}_K = \dfrac{(\hat{f_\sigma})}{\nu|K|^2}$ ✓ $\Rightarrow$ $K\cdot\hat{u}_K = \dfrac{(\hat{f_\sigma})_K}{\nu|K|^2}\cdot K = 0$ $\rightarrow \mathrm{div}(f_\sigma) = 0$

$\Rightarrow \hat{u}_K = \dfrac{(\hat{f_\sigma})_K}{\nu|K|^2} \rightarrow$ Conclude:

$\dfrac{4}{L^2}\sum_K |K|^2|\hat{u}_K|^2 = \sum_K \dfrac{|\hat{f_\sigma}|^2}{\nu^2|K|^2} \le \dfrac{C(L)}{\nu^2}\|f_\sigma\|^2_{H^{-1}}$

$\dfrac{1}{L^2}\sum_K |K|^2|\hat{u}_K|^2 \le \dfrac{C(L)^2}{L^2\nu^2}\|f_\sigma\|^2_{H^{-1}}$

$\Rightarrow u \in \dot{H}^1$ & $\|u\| \le \tilde{C}(L)\cdot \frac{1}{\nu}\|f_\sigma\|_{H^{-1}}$. $\Rightarrow$ $u \in \dot{H}^1 \rightarrow$ Regularity

Projection
$\downarrow$

$\Rightarrow \|u\|_{H^1} \leq \frac{\tilde{C}(L)}{\nu} \|f\|_{H^{-1}}, \quad u = \sum \hat{\mu}_k e^{\frac{2\pi i k \cdot x}{L}}.$

Done ☺.

- If $f \in \dot{L}^2_{per}(\Omega)$. Repeat! Apply decomposition.

$f_\sigma \in \dot{L}^2_{per}(\Omega), \; g \in \dot{H}^1$

Then $p \in \dot{H}^1, \; \sum |k|^4 |\hat{u}_k|^2 = \sum |\hat{f}_{\sigma k}|^2 \leq |f|_{L^2}$

$\Rightarrow u \in H^2 \cap V, \; \|u\|_{H^2} \leq C \|f\|_{L^2}.$

$V \subset H, \; V', H' \to$ Duals.
$H' \to$ Riesz H

$$\boxed{V \underset{\text{compact}}{\hookrightarrow} H = H' \underset{\text{compact}}{\hookrightarrow} V'}.$$

Summary: $\begin{cases} -\nu\Delta u + \nabla p = f \\ \operatorname{div}(u) = 0 \end{cases}$ → might as well take $f = f_\sigma$.

Can show $\|u\|_{H^2} \leq C(L,\nu)\|f\|_{L^2} = C(L,\nu)|f|.$
solution

$H \xrightarrow[1\text{-}1]{S} H^2 \cap V \xrightarrow[\text{compact}]{i} H$ → Solution map.
$f \longmapsto u$ $\qquad f \longmapsto u$

$S : H \underset{1\text{-}1}{\longrightarrow} H$ is compact, linear bounded.

Let $v \in H^2 \cap V, \; -\nu\Delta v =: g \in H$ (L^2 w/ 0-div.) so S: $H \xrightarrow[\text{onto}]{1\text{-}1} H^2 \cap V$.

$\boxed{\nu = 1}$

Properties of S

i: $S: H \to H$, (1-1), (ii) compact

iii: S is symmetric, let $f, g \in H$

$(Sf, g) = (f, Sg)$.

Proof: $-\Delta u = f$, $-\Delta v = g$. S is self-adjoint.

$$\underset{(Sf,g)}{\underbrace{(u, -\Delta v)}} = (\nabla u, \nabla v) = (\nabla v, \nabla u) = \underset{(Sg,f)}{\underbrace{(v, -\Delta u)}}$$

$(Sf, f) = (\nabla u, \nabla u) = \|\nabla u\|_{L^2}^2 \geq 0 \Rightarrow S$ is non-neg operator.

Theorem: $\exists \{\varphi_k\}$ o.n. basis of H of eigenfunctions of S, ie, $S\varphi_k = \mu_k \varphi_k$; spectrum is discrete, $0 < \cdots < \mu_k \ldots \leq \mu_2 \leq \mu_1$.

S^{-1} (exists) $= A \to$ Stokes Operator, $u = Sf$.

$-\Delta u = f = S$, $Au = f$. $\to \boxed{D(A) = H^2 \cap V}$

$A: D(A) \cap V \to H$, $A\varphi_k = \frac{1}{\mu_k}\varphi_k$, $\mu_k = \lambda_k$ $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \to \infty$,

$$\Rightarrow \begin{cases} -\Delta \varphi_k = \lambda_k \varphi_k \\ \operatorname{div}(\varphi_k) = 0 \end{cases}$$

$A: D(A) = H^2 \cap V \to H$. Claim: $\mathcal{V} \subset D(A)$, dense in H^2. $|Au| = |-\Delta u| \lesssim \|u\|_{H^2}$

IBP

$\to$ Extension of $A: V \to V'$

Au, $\{\psi_k\}$, $\{\varphi_n\} \subset \mathcal{V}$ $(A\varphi_n, \psi_k) = (-\Delta\varphi_n, \psi_k) \stackrel{\downarrow}{=}$

$(\nabla\varphi_n, \nabla\psi_k)$. Let $\psi \in V$, $\varphi_n \in \mathcal{V}$, $\psi_k \to \psi$ in V

$$|\underbrace{(A\varphi_n, \psi_k)}| = |(\nabla\varphi_n, \nabla\psi_k)| \le |\nabla\varphi_n|\,|\nabla\psi_k| = \|\varphi_n\|\,\|\psi_k\|$$

$$|\langle A\varphi_n, \psi_k\rangle_{V',V}| \le \|\varphi_n\|\cdot\|\psi_k\| \Rightarrow |\langle A\varphi_n, \psi\rangle_{V',V}| \le \|\varphi_n\|\cdot\|\psi\|.$$

$$\Rightarrow \|A\varphi_n\|_{V'} \le \|\varphi_n\|.$$

Let $\varphi_n \to \varphi$ in V. $\|A\varphi_n - A\varphi_m\|_{V'} \le \|\varphi_n - \varphi_m\| \to 0$.

$\Rightarrow A\varphi_n$ is Cauchy in $V' \Rightarrow \exists\, w \in V'$ s.t.
$A\varphi_n \to w$ in V'. $\boxed{\text{Call } w = A\varphi}$

$\varphi, \psi \in V$.

Conclude $\langle A\varphi, \psi\rangle_{V'\times V} = ((\varphi, \psi))$

$\Rightarrow \langle A\varphi, \varphi\rangle_{V'\times V} = \|\varphi\|^2$ and $\|A\varphi\|_{V'} = \|\varphi\|$.

L^2

$A\varphi_k = \lambda_k \varphi_k \to A^\alpha \varphi_k = \lambda_k^\alpha \varphi_k$, $u \in H$ $|u|^2 = \sum (u, \varphi_k)^2$.

Note $u_N = \sum_{k=1}^{N} (u, \varphi_k)\varphi_k$, $|u_N - u|^2 = \sum_{k>N} (u, \varphi_k)^2 \xrightarrow{N\to\infty} 0$.

$\alpha \ge 0$, $A^\alpha u_N = \sum_1^N (u, \varphi_k)\lambda_k^\alpha \varphi_k$ $|A^\alpha u_N|^2 = \sum_1^N \lambda_k^{2\alpha} |(u, \varphi_k)|^2$

$D(A^\alpha) := \{u \in H : \sum_k |(u, \varphi_k)|^2 \lambda_k^{2\alpha} < \infty$

$\boxed{\text{claim}}$

<u>Ex:</u> $D(A^{1/2}) = \{u \in H : \sum |(u, \varphi_k)|^2 \lambda_k < \infty\} = V$.

Let $u_N \to u$ in V; $u_N := \sum_{k=1}^{N} (u, \varphi_k)\varphi_k$. Then,

$$\|u - u_N\|^2 = \langle A(u_N - u), u_N - u \rangle_{V',V}$$

$$= \sum_{k=1}^{N} (u, \varphi_k)^2 \lambda_k - \|u\|^2 \to 0.$$

Let $u \in V$, $\underset{H^1}{\|u\|^2} = \sum (u, \varphi_k)^2 \lambda_k \quad 0 < \lambda_1 \le \lambda_2 \le \dots \lambda_k \to \infty.$

$$\underbrace{\ge \lambda_1 \sum_{k=1}^{\infty} (u, \varphi_k)^2 = \lambda_1 |u|^2.}_{}$$

Poincaré! Inequality!

Feb-4-2025

<u>Recap</u>: $-\nu\Delta u + \nabla p = f$ u-periodic on $\Omega = (\mathbb{T})^d$
$\mathrm{div}(u) = 0$

$\Leftrightarrow$ $f_\sigma + f_g$ $\nearrow f \in H$, $u \in V$

$\Leftrightarrow$ $-\nu\Delta u = f$, $\nabla p = \nabla g \rightarrow p = g$.
$\mathrm{div}(u) = 0$

$u = S(f) \rightarrow$ solution operator.

$H \xrightarrow{S} V$, $\|u\|_V \le C|f|$, $f \in H \Rightarrow u \in \dot{H}^2 \cap V$

$S: H \xrightarrow{\text{onto}} \dot{H}^2 \cap V \underset{\text{compact}}{\hookrightarrow} H$

$S: H \rightarrow H$ compact

S has eigenfunctions, $S\varphi_k = \mu_k \varphi_k$. FD.

$A = S^{-1}$, $A: \underset{\dot{H}^2 \cap V}{D(A)} \rightarrow H$. $\rightarrow$ Stokes Operator, $\underset{?}{\mathbb{P}_\sigma}(\Delta^{+})^{-2}$ j^n

$A\varphi_k = \lambda_k \varphi_k$, $\lambda_k = \dfrac{1}{\mu_k}$, $0 < \lambda_1 < \lambda_2 < \dots$ $\boxed{\lambda_k \underset{k \to \infty}{\sim} k^{d/2} \lambda_1}$

{Weyl, Meterriez}

$\tilde{A} = -\Delta$, $-\Delta u + \nabla p = f$.

$-P_\sigma(\Delta u) + \underbrace{P_\sigma(\nabla p)}_{0} = P_\sigma(f) = f_\sigma$

$A = (-\mathbb{P}_\sigma \Delta)$. $\rightarrow$ <u>Non-local</u>, so "more" than Laplace.

$\langle Au, v \rangle_{V' \times V} = ((u, v))$, $A: V \rightarrow V'$.

$\alpha \geq 0$, $D(A^{\alpha}) = \{u \in H : \sum_{k=1}^{\infty} |(u, \varphi_k)|^2 \lambda_k^{2\alpha} < \infty\}$,

$|A^{\alpha} u| = \sum |(u, \varphi_k)|^2 \lambda_k^{2\alpha}$, $A\varphi_k = \lambda_k \varphi_k$

$D(A^{1/2}) = V$, $\|u\|_V = |A^{1/2} u|_V$ → Exercise. Wyl's give rate.

$\alpha > 0$, $D(A^{-\alpha}) = \{\mu \in \dot{D}(\Omega)' : \sum_k |\langle \mu, \varphi_k \rangle|^2 \frac{1}{\lambda^{2\alpha}} < \infty\}$

$(D(A^{\alpha}))' = D(A^{-\alpha})$, $\langle \mu, w \rangle = \sum_1^{\infty} \hat{\mu}_k \hat{w}_k$

$$= \sum_1^{\infty} \frac{\langle \mu, \varphi_k \rangle}{\lambda_k^{\alpha}} (w, \varphi_k) \lambda_k^{\alpha}$$

$(D(A^{1/2}))' = D(A^{-1/2}) = V'$, $\|u\|_{V'} = |A^{-1/2}$

<u>Remark</u> $\mu \in V'$. Then $\exists\, \tilde{\mu} \in \dot{D}(\Omega)'$ s.t.
$\langle \tilde{\mu}, \varphi \rangle := \langle \mu, \varphi \rangle_{V' \times V}$ $\forall \varphi \in \mathcal{V} \subset V$

Let $\varphi_n \to \varphi$ in $\dot{D}(\Omega)$, $\varphi_n \to \varphi$ in V. So convergence holds. ~~Is~~ extension unique?[2]
$\langle \tilde{\mu}_1 - \tilde{\mu}_2, \varphi \rangle = \langle \tilde{\mu}_1, \varphi \rangle_{V \times V'} - \langle \tilde{\mu}_2, \varphi \rangle_{V \times V'} = 0$ $\forall \varphi \in \mathcal{V}$. So by De Rham, $\exists\, \nabla \pi$ s.t. $\tilde{\mu}_1 - \tilde{\mu}_2 = \nabla \pi$.

$\xrightarrow{\text{Msbl.}}$

Bochner: (X, Σ, μ), $\mathcal{B}$-Banach,
$f: X \to B$ is measurable if $f^{-1}(U) \in \Sigma$ $\forall U$ open in $\mathcal{B}$.

Bochner-Measurable: $\exists$ $S_n(x)$ simple functions s.t. $S_n(x) \longrightarrow f(x)$ in $\mathcal{B}$ a.e.

$\{E_j\} \subset \Sigma$, $b_j \in \mathcal{B}$, $E_j \cap E_k = \emptyset$, $S_n(x) = \sum_1^n b_j \chi_{E_j}(x)$.

$$\int s_n(x)\,dx = \sum_1^n b_j \mu(E_j) \in \mathcal{B}.$$

Def: f is Bochner-integrable if f is B-msbl as $\lim_{n\to\infty} \int_X \|s_n(x) - f(x)\|_{\mathcal{B}}\, d\mu \to 0$.

Define: $\int_X f(x)\,d\mu = \lim_{n\to\infty} \int_X s_n(x)\,d\mu$. Why exists? Cauchy Sequence

If f is B-measurable, then f is integrable iff $\int_X \|f\|_{\mathcal{B}}\, d\mu < \infty$.

Ex: Let $T \in \mathcal{B}'$, f B-msbl. $\Rightarrow$ $\langle T, f\rangle$ is msbl.
Then:
$$\langle T, \int f\,d\mu\rangle = \int_X \langle T, f\rangle\, d\mu.$$

Ex: Let $x, y \in \mathcal{B}$, $\underbrace{\langle T, x\rangle = \langle T, y\rangle}$ $\forall T \in \mathcal{B}'$, $\Leftrightarrow$ $x = y$.

$\boxed{\text{Functionals are like coordinates in Banach Space!}}$

Temam Book

Lemma: Let X be a Banach space, X' dual.
Let $u, g \in L^1((a,b), X)$. TFAE:

1) $\exists\ \xi \in X$ s.t. $u(t) = \xi + \int_a^t g(s)\,ds \quad \forall$ ~~a.e.~~ all $t \in [a,b]$

$\hookrightarrow$ u - abs cont.

$u' = g$

2) $\forall\ \varphi \in \mathcal{D}(a,b)$, we have

$$\int_a^b u(t)\,\varphi'(t)\,dt = -\int_a^b g(t)\,\varphi(t)\,dt.$$

3) $\forall \eta \in X'$, $\frac{d}{dt}\langle \eta, u(t)\rangle_{X' \times X} = -\int_a^b \langle \eta, g(t)\rangle\,dt.$

Feb.-11-25

$u \in C^1([0,T],H)$, $|u(t)|^2 \in C^1[0,T]$,

$$\frac{d}{dt}(u(t),u(t)) = (u'(t),u(t)) + (u(t),u'(t)) = 2(u'(t),u(t)).$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}|u(t)|^2 = u'(t)u(t).$$

Suppose, in addition $u \in C([0,T],V)$, $u'(t) \in C([0,T],H) \subset C([0,T],V')$. $H \subset V'$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}|u(t)|^2 = \langle u'(t), u(t)\rangle_{V'\times V}.$$

<u>Lemma</u>: Lions-Magenes: Suppose $u' \in L^2([0,T],V')$.

Suppose $u \in L^2((0,T),V)$, $|\langle u'(t), u(t)\rangle|_{V'\times V}$

$\leq \|u'(t)\|_{V'} \|u(t)\|_V \Rightarrow \int_0^T |\langle u'(t), u(t)\rangle| dt < \infty$

so $\langle u'(t), u(t)\rangle \in L^1([0,T])$. Then, $|u(t)|^2$ is absolutely cont. and $\frac{d|u|^2}{dt} = 2\langle u', u\rangle_{V'\times V}$ a.e.

<u>Proof</u>: Apply mollifiers.

<u>Ex</u>: Let $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$. Suppose X = Banach, X^* = dual. $(L^p([0,T],X))' = L^{p'}([0,T],X')$.

$\boxed{\text{Generalizations}}$ $\quad u \in \left(L^p((0,T), V) \right)' = L^{p'}((0,T), V')$

moreover, if X = Hilbert, $\langle u'(t), u(t) \rangle_{V' \times V} = \frac{1}{2}\frac{d}{dt}|u(t)|^2$.

Steady-State Navier-Stokes

$$\begin{cases} -\nu\Delta u + (u\cdot\nabla)u + \nabla p = f, & f \in L^2(\Omega), \text{ given}, \\ \operatorname{div}(u) = 0 & f = f_\sigma + f_g = \mathbb{P}_\sigma(f) + \nabla g. \end{cases}$$

$$\Rightarrow \begin{cases} -\nu\Delta u + (u\cdot\nabla)u + \nabla p = \mathbb{P}_\sigma(f) + \nabla g. \\ \operatorname{div}(u) = 0 \end{cases}$$

$$-\nu\,\mathbb{P}_\sigma(\Delta u) + \mathbb{P}_\sigma((u\cdot\nabla u)u) \underbrace{-\nu(I-\mathbb{P}_\sigma)(\Delta u)}_{0} + (I-\mathbb{P}_\sigma)(u\cdot\nabla u)u + \nabla p = f_\sigma + \nabla g.$$

$$-\nu\,\mathbb{P}_\sigma(\Delta u) + \mathbb{P}_\sigma((u\cdot\nabla u)u) + (I-\mathbb{P}_\sigma)(u\cdot\nabla u)u + \nabla p = f_\sigma + \nabla g$$

$$\Rightarrow \begin{cases} -\nu\,\mathbb{P}_\sigma(\Delta u) + \mathbb{P}_\sigma((u\cdot\nabla u)u) = f_\sigma \xrightarrow{\text{Only } u!} \text{System} \\ (I-\mathbb{P}_\sigma)((u\cdot\nabla u)u) + \nabla p = \nabla g. \end{cases}$$

$$-\nu\,\underbrace{\mathbb{P}_\sigma(\Delta u)}_{\text{Stokes op.}} + \mathbb{P}_\sigma((u\cdot\nabla u)u) = \mathbb{P}_\sigma(f) \rightarrow \boxed{\text{Solve first}}$$

$$\nu A u + \mathbb{P}_\sigma((u\cdot\nabla u)u) = \mathbb{P}_\sigma(f).$$

Consider $\displaystyle\int (u\cdot\nabla)v\cdot w\,dx \leq \int \underbrace{|u|}_{L^3}\underbrace{|\nabla v|}_{L^2}\underbrace{|w|}_{L^6 \to SET}dx$, $\quad u, v, w \in H$

$$\leq \|u\|_{L^2}\|\nabla v\|_{L^2}\|w\|_{L^6} \leq$$

Defines a functional,

$C\|u\|_{H^{1/2}} \|u\| \cdot \|w\| \Rightarrow$

$\int (u\cdot\nabla)v\cdot w\, dx \le C\, |u|^{1/2} \|u\|^{1/2} \|v\| \|w\|$

$B(u,v) \in V'$, $\langle B(u,v), w\rangle$,

$\|B(u,v)\|_{V'} \le C |u|^{1/2} \|u\|^{1/2} \|v\|$.

$B: V \times V \to V'$, $\|B(u,v)\|_{V'} \le C |u|^{1/2} \|u\|^{1/2} \|v\|$

$w = \mathbb{P}_\sigma w \Rightarrow$

$$\int (u\cdot\nabla)v\cdot w\, dx = \int (u\cdot\nabla)v \cdot \mathbb{P}_\sigma(w)\, dx \quad \boxed{?!}$$

$$= \int \mathbb{P}_\sigma(u\cdot\nabla v)\cdot w\, dx$$

$\Rightarrow \quad B(u,v) = \mathbb{P}_\sigma((u\cdot\nabla)v)$, $B(u,u) = \mathbb{P}_\sigma(u\cdot\nabla u)$, $u \in V$.

Q: $\underset{V'}{\nu A u} + \underset{V'}{B(u,u)} = \underset{V'}{\mathbb{P}_\sigma(f)}$ in V', $u \in V$. $f \in L^2 \Rightarrow \mathbb{P}_\sigma(f) \in H \subset V'$.

Summary: Let $f \in L^2(\Omega)$. Soln $\begin{cases} -\nu\Delta u + (u\cdot\nabla)u = f_\sigma \\ \operatorname{div}(u) = 0 \end{cases}$

$\Longleftrightarrow \begin{cases} \nu A u + B(u,u) = f_\sigma \text{ Find } u \in V. \\ (I - \mathbb{P}_\sigma)u + \nabla p = \nabla q \end{cases}$

Take $f = f_0$, so $\nabla q = 0$.

$$\nu \overset{\nearrow V'}{A u} + \overset{\nearrow V'}{B(u,u)} = f \quad \nearrow \text{Solve this.}$$

Focus: Let $f \in H$. Find $u \in V$ s.t. eq holds in V'.

a priori estimates: Let $u \in V$ be a solution.

$$\langle (\nu A u + B(u,u) - f), u \rangle_{V' \times V} = 0.$$

$$= \nu \|u\|^2 + \langle B(u,u), u \rangle_{V' \times V} = \langle f, u \rangle_{V' \times V}$$

$\nearrow$ Ex:

<u>Lemma</u>: $\langle B(u,v), w \rangle_{V' \times V} = \int (u \cdot \nabla) v \cdot w \, dx$, consider

$\mathcal{V} \subset V$, div-free trig polys, $\overline{\mathcal{V}} = V$.

$u, v, w \in \mathcal{V}$

$$\langle B(u,v), w \rangle = \int (u \cdot \nabla) v \cdot w \, dx = \sum_{k,l=1}^{d} \int \underbrace{u_k \frac{\partial}{\partial x_k}}_{= \frac{\partial}{\partial x_k}(u_k v_l) \text{ by div-free}} v_l \, w_l \, dx$$

$$= -\sum_{k,l=1}^{d} \int u_k v_l \frac{\partial}{\partial x_k} w_l \, dx = -\langle B(u,w), v \rangle_{V' \times V}. \quad \checkmark$$

$\forall u, w, v \in V$.

$$\Rightarrow \langle B(u,v), w \rangle_{V' \times V} = -\langle B(u,w), v \rangle_{V' \times V}.$$

<u>Cor:</u> $\langle B(u,v), v \rangle = 0$, so $(B(u,u), u) = 0$. $\blacksquare$

Back to a-priori: $\nu \|u\|^2 = \langle f, u \rangle_{V' \times V} = (f, u) \le |f| |u| \le |f| \cdot \frac{\|u\|}{\lambda_1^{1/2}}$.

$\Rightarrow \quad \|u\| \le \dfrac{|f|}{\nu \lambda_1^{1/2}} := R_1.$

$\to$ Now go discrete. $A\varphi_k = \lambda_k \varphi_k$.

$$u = \sum_{k=1}^{\infty} \tilde{a}_k \varphi_k$$

Formally, $\sum_1^{\infty} \nu \lambda_k \tilde{a}_k \varphi_k + B\left(\sum_1^{\infty} \tilde{a}_k \varphi_k, \sum_1^{\infty} \tilde{a}_l \varphi_l\right) = \sum_1^{\infty} f_k \varphi_k.$

$$\nu \sum_{k=1}^{\infty} \tilde{a}_k \lambda_k \varphi_k + \sum_{k,l=1}^{\infty} \tilde{a}_k \tilde{a}_l B(\varphi_k, \varphi_l) = \sum_1^{\infty} f_k \varphi_k, \quad \varphi_j \text{ fixed.}$$

$$\nu \tilde{a}_j \lambda_j + \sum_{k,l=1}^{\infty} \tilde{a}_k \tilde{a}_l \underbrace{\langle B(\varphi_k, \varphi_l), \varphi_j \rangle}_{c_{klj}} = f_j \quad \forall j = 1 \ldots \infty.$$

$$(B(\varphi_k, \varphi_l), \varphi_j) = \int (\varphi_k \cdot \nabla) \varphi_l \cdot \varphi_j \, dx.$$

Solve This.

$\boxed{\text{Feb-13-2025}}$

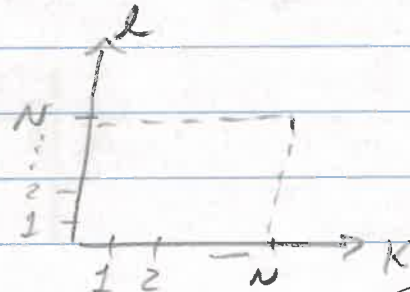
$\to f \in H$; $\nu Au + B(u,u) = f$, find $u \in V$ s.t holds, as an element in V', i.e. $\forall v \in V$.

$$(\nu Au + B(u,u), v)_{V' \times V} = (f, v)_{V' \times V} = (f, v)$$

Formally, seek $u = \sum \hat{a}_k \varphi_k$.

$\iff$

$$\nu \lambda_j \hat{a}_j + \sum_{k,l=1}^{\infty} c_{klj} \hat{a}_k \hat{a}_l = f_j, \quad c_{klj} = \langle B(\varphi_k, \varphi_l), \varphi_j \rangle_{V' \times V}$$



Rest of sum;

$$\nu \lambda_j \hat{a}_j + \sum_{k,l=1}^{N} c_{klj} \hat{a}_k \hat{a}_l + \boxed{\quad} = f_j \quad \forall j \in \mathbb{N}.$$

$j = 1, \ldots, N$

approximate by killing tail:

$$\nu \lambda_j \hat{a}_j + \sum_{k,l=1}^{N} c_{klj} \hat{a}_k \hat{a}_l \approx f_j, \text{ so get}$$

$$\underbrace{\nu \lambda_j a_j + \sum_{k,l=1}^{N} c_{klj} a_k a_l = f_j}_{\text{Galerkin Approximation}} \longrightarrow j = 1, \ldots, N \text{ solve this finite system.}$$

$\boxed{\text{Galerkin Approximation.}}$

→ Eigenfunctions.

Define $H_n = \text{span}\{\varphi_1, \to \varphi_n\}$.

$$\sum_{j=1}^{N}\left(\nu \lambda_j a_j \varphi_j + \sum_{k,l=1}^{N} \langle B(\varphi_k, \varphi_l), \varphi_j \rangle a_k a_l \varphi_j\right) = \sum_{j=1}^{N} f_j \varphi_j$$

$= P_N(f) \to$ Projection

$\Downarrow$

$$\nu A\Big(\underbrace{\sum_{j=1}^{N} a_j \varphi_j}_{u_N \in H_n}\Big) + \sum_{j=1}^{N} (B(u_N, u_N), \varphi_j)\varphi_j = P_N(f)$$

$$w = \sum_{j=1}^{N} (w, \varphi_j)\varphi_j = P_N(w) \Rightarrow$$

(*) $\boxed{\nu A u_N + P_N(B(u_N, u_N)) = P_N(f).}$

Since eigenfunction are C^∞, u_N is C^∞.

Galerkin approx $\Leftrightarrow$ to

$$\nu A u_N + P_N B(u_N, u_N) = P_N(f).$$

↳ So equation holds in classical sense.

$\boxed{H^n \cong \mathbb{R}^n}$ Q: Do we have a solution to (*)?

Euclidean

Proposition: Let $\Phi: \mathbb{R}^n \to \mathbb{R}^n$, continuous, $(\mathbb{R}^n, \langle \cdot, \cdot \rangle_E)$. Let $R > 0$ given s.t. $\langle \Phi(v), v \rangle < 0$ $\forall v \in \mathbb{R}^n$ w/ $\|v\| = R$. Then, $\exists v^* \in B(0,R)$ s.t. $\Phi(v^*) = 0$.

$\to$ open ball

Proof: Let $B(R,0)$, $\phi: \overline{B^{(0)}} \to \mathbb{R}^n$ continuous.
Suppose $\phi(v) \neq 0 \ \forall v \in \overline{B}(0,R)$. Define
$F = \frac{\phi(v)}{\|\phi(v)\|} \cdot R$. $F: \overline{B(R,0)} \to \underset{R}{\overline{B}}(0)$, F is continuous.

By Brouwer fixed point, $\exists\ \bar{v} \in \overline{B}(0,R)$ s.t.
$F(\bar{v}) = \bar{v}$. But $\|F(\bar{v})\| = \|\bar{v}\| = R$, so $\|\bar{v}\| = R$ ie,
$\bar{v} \in \partial\overline{B(0,R)}$. Then, $\langle F(\bar{v}), \bar{v}\rangle = \|\bar{v}\|^2 = R^2$,
or $\left\langle \frac{\phi(\bar{v})}{\|\phi(\bar{v})\|}, \bar{v} \right\rangle R = R^2 \Rightarrow$ C. If, $< \to \leq$ in statement,
then $R=0 \Rightarrow$ C.

Back to Galerkin:

$$\nu A u_N + P_N B(u_N, u_N) = P_N(f). \quad (*)$$

Find a priori estimates on the solutions, take L^2-inner prod of u_N: $\boxed{P_N \text{ is self adjoint}}$

$$\nu(Au_N, u_N) + P_N(B(u_N,u_N), u_N) = (P_N f, u_N).$$

$$\nu\|u_N\|^2_{H^1} + \underbrace{(B(u_N,u_N), \overset{u_N}{\overset{\|}{P u_N}})}_{0} = (f, u_N)$$

$$\leq \frac{|f|}{\lambda^{1/2}} \cdot \|u_N\| =: R^2.$$

use V-norm:

$$\Phi(v) = -v - (\nu A)^{-1} P_N(B(v,v)) + (\nu A^{-1}) P_N(f), \quad v \in H^n.$$

$\to \Phi$ is continuous b.c. system is system quadratic in coefficients.

Let $\boxed{v \in H_N,}$ $\|v\| = R_1$,

$$((\Phi(v), v)) = (\tfrac{\nu}{\nu} A\, \Phi(v), v) = \frac{1}{\nu}[-\|v\|^2 - \underbrace{(P_N(B(v,v), v)}_{0} + (P_N(f), v)].$$

H^1-i.p.

$$\Rightarrow ((\Phi(v), v)) = -\frac{1}{\nu}(\nu\|v\|^2 - (f, v)).$$

$$\le \frac{1}{\nu}(-\nu\|v\|^2 + |f|\,|v|)$$

$$\le \frac{1}{\nu}\left(-\nu\|v\|^2 + \frac{|f| \cdot \|v\|}{\lambda_1^{1/2}}\right)$$

$$\le \frac{1}{\nu}\left(-\nu\frac{|f|^2}{\nu^2\lambda_1} + |f|\frac{|f|}{\nu\lambda_1^{1/2}\lambda_1^{1/2}}\right) = 0.$$

Thus $\exists\, v^* \in H_N$, $\|v^*\| \le R_1$ s.t.
$\Phi(v^*) = 0$. Solution exists. ∎

Summary: $\nu A u_N + P_N B(u_N, u_N) = P_N f$ has at least 1 solution u_N and all solutions satisfy $\|u_N\| \le R_1$. R_1 is N-independent, so $\{u_N\}$ is a sequence in V uniformly bounded, $\{u_N\} \subset V \underset{\text{comp.}}{\hookrightarrow} H$,

$\exists\, u_{N_j} \xrightarrow{\text{weakly}} u$ in V

$u_{N_j} \underset{\text{str.}}{\longrightarrow} u$ in H.

Feb. 20-25

Theorem: (Simon's Compactness)
$F \subset L^p(0,T;X)$

- F is precompact in $L^p(0,T;X)$ $p<\infty$ or $C(0,T;X)$ for $p=\infty$

$\Leftrightarrow \begin{cases} 1) \left\{\int_{t_1}^{t_2} f(t)dt, f \in F\right\} \text{ is precompact in } X \\ 2) \text{ Equicontinuity.} \end{cases}$

Corollary: (Aubin-Lion's compactness)
assume $X_0 \underset{comp}{\subset\subset} X \underset{cont}{\subset} X_1$ w/ $X_0 \xrightarrow{id} X$ is compact. Set $F = \{v \in L^p(0,T;X_0) : v' \in L^q(0,T;X_1)\}$, $1 \le p < q \le \infty$. Then

1.) If $p<\infty$, then F is precompact in $L^p(0,T;X)$ 2) If $p=\infty$, F is precompact in $C(0,T;X)$.

Eg: N-S: $X_0 = V = \{v \in H^1 \text{ and } div(v) = 0\}$
$X = H = \{v \in L^2 \text{ and } div\, v = 0\}$
$\int v \cdot \nabla \varphi = 0$
$X_1 = (V)'$.

Lemma: $X_0 \subset\subset X \subset X_1$ $\forall \eta > 0$, $\exists C_\eta \in (0, \infty)$ s.t. $\|f\|_X \le \eta \|f\|_{X_0} + C_\eta \|f\|_{X_1}$ $\forall f$ s.t. RHS is finite

Rk: $X = L^2$, $X_0 = H_1^0$, $X_1 = L^1$, $D \subset\subset \mathbb{R}^n$.

G-N inequality, $\|f\|_2 \le \|f\|_{L^{p*}}^{\theta} \|f\|_{L^1}^{(1-\theta)}$ $0 < \theta < 1$.

$\le C \|f\|_{H_0^1}^{\theta} \|f\|_{L^1}^{1-\theta}$

Young. $\rightarrow$ $\le \eta \|f\|_{H^1} + \left(\frac{C}{\eta}\right)^{\frac{1}{1-\theta}} \|f\|_{L^1}$.

Proof of Lemma: By contradiction. Otherwise $\forall K \to \infty$, $\exists f_K$ $\|f_K\|_X \ge \eta \|f_K\|_{X_0} + K \|f_K\|_{X_1}$. Assume wlog

$\|f_K\|_{X_0} = 1$. $\Rightarrow$ $\eta + K\|f_K\|_{X_1} \le \|f_K\|_X \le \|f_K\|_{X_0} = 1$.

$\Rightarrow$ $K \to \infty$, $\|f_K\|_{X_1} \to 0$. So we have bounded

sequence in X_0, so exists a strongly convergent subsequence in X. But that subsequence must go to 0, and so to 0 in X_0. $\Rightarrow$ C

Proof of Aubin Only verify 1 & 2: 1 is

trivial, $X_0 \subset\subset X$, $\|\int_{t_1}^{t_2} u(t)\,dt\|_X \le \|\int_{t_1}^{t_2} u(t)\,dt\|_{X_0}$

$\le \int_{t_1}^{t_2} \|u(t)\|_{X_0}\,dt \le (t_2 - t_1)^{1-1/p} \left(\int_{t_1}^{t_2} \|u(t)\|_{X_0}^p\,dt\right)^{1/p} < \infty$.

ie, $\int_{t_1}^{t_2} u(t)\,dt$ is bounded in $X_0 \subset\subset X$.

To show (2), $\forall \eta$, C_η as in lemma

$$\|\tau_h u - u\|^p_{L^p(0,T;X)} = \int_0^T \|\tau_h u - u\|_X^p \, dt$$

$$\leq \int_0^T \eta^p \|\tau_h u - u\|_{X_0}^p \, dt + \int_0^T C_\eta^p \|\tau_h u - u\|_{X_1}^p \, dt$$

$\tau_h(u) = u(t+h)$, $\tau_h(u) - u(t) = \int_t^{t+h} u'(s) \, ds$

$$\leq \eta^p C_p \int_0^T \|\tau_h u - u\|_{X_0}^p \, dt + C_p C_\eta^p \int_0^T \Big\| \int_t^{t+h} u'(s) \, ds \Big\|_{X_1}^p \, dt$$

$$\leq \eta^p \cdots + C_p C_\eta^p \int_0^T \Big(\int_t^{t+h} \|u'(s)\|_{X_1}^b \, ds\Big)^{p/b} (h)^{p(1-\frac{1}{b})} \, dt.$$

$$\leq \eta^p C_p \|u\|_{L^p(0,T;X_0)} + h^{p(1-\frac{1}{b})} C_p C_\eta^p \|u'\|_{L^p(0,T;X_1)}$$

$$\Rightarrow \leq C_1 \eta^p + h^{p(1-\frac{1}{b})} C_\eta^p C, \text{ so choose } \eta, h \to 0.$$

∎

Proof of Simon's Theorem:

$1: \Rightarrow$ Assume F pre-compact in $L^p(0,T;X)$, $1 \leq p < \infty$

$f \in F$, $\int_{t_1}^{t_2} f(t) \, dt \in X$,

$$\underbrace{f \longrightarrow \int_{t_1}^{t_2} f(t) \, dt}_{\text{cont.}}, \quad L^p(0,T;X) \to X$$

$\Longrightarrow 1.$

To show (2): $\forall \varepsilon > 0$ $\exists$ a finite number of $f_i \in L^p(0,T;X)$, $1 \le i \le N_\varepsilon$, $\forall f \in F$,

$$\|f - f_i\|_{L^p(0,T;X)} \le \varepsilon/3.$$

$$\|f_i - f_j\|_{L^p(0,T;X)} > \varepsilon/4$$

Indeed if $N_\varepsilon = \infty$, then $\{f_i\}$ is bounded in $L^p(0,T;X)$ and no Cauchy sequence $\Rightarrow$ C since F is compact.

As $C(0,T;X)$ is dense in $L^p(0,T;X)$,

$\exists$ $\hat{f}_i$ s.t. $\hat{f}_i \in C(0,T;X)$ and

$$\|\hat{f}_i - f_i\|_{L^p(0,T;X)} \le \varepsilon/3, \quad \exists i, \exists h_i \text{ s.t.}$$

$\forall h < h_i$

$$\|\tau_h \hat{f}_i - \hat{f}_i\|_{L^p(0,T-h,X)} \le \varepsilon/3.$$

$\eta = \min_{0 < i \le N_\varepsilon} h_i$ $\quad$ $\forall f \in F$, $\exists i$ s.t. $\tau_h f - f = \tau_h f - \tau_h f_i + \tau_h f_i - \tau_h \hat{f}_i + \tau_h \hat{f}_i - \hat{f}_i + \hat{f}_i - f_i + f_i - f$

$$\Rightarrow \|\tau_h f - f\|_{L^p(0,T-h,X)} \le \|\tau_h f - \tau_h f_i\|_{L^p(0,T-h,X)} + \|\tau_h f_i - \tau_h \hat{f}_i\|_{L^p(X)} + \|\tau_h \hat{f}_i - \hat{f}_i\|_{L^p(X)} + \|\hat{f}_i - f_i\|_{L^p(X)} + \|f_i - f\|_{L^p(X)} \le \varepsilon \text{ after some } \varepsilon\text{-magic.}$$

This is $\longleftarrow$
2.

2) $\Leftarrow$ Assuming F sat. 1&2

2.1 Claim: The local mean is precompact in $L^p(0,T;X)$.

Indeed, for $f \in F$ $a>0$,
Set right mean function
$$(M_a f)(t) := \frac{1}{a}\int_t^{t+a} f(s)\,ds \in C(0,T,X).$$

$(M_a)f := \{M_a f : \forall f \in F\}$ bounded in $C(0,T-a;X)$.

$\forall\ 0 \le t_1 \le t_2 \le T-a$,
$$\underbrace{\| M_a f(t_2) - M_a f(t_1)\|_X}_{*} = \left\| \frac{1}{a}\left(\int_{t_1}^{t_2+a} f(s)\,ds - \int_{t_1}^{t_1+a} f(s)\,ds\right)\right\|_X$$
$$= \frac{1}{a}\left\| \int_{t_1}^{t_1+a} \tau_{t_1-t_2} f(s) - f(s)\,ds \right\|_X$$
$$* \le \frac{1}{a} \| \tau_{t_2-t_1} f - f\|_{L^1(0,T-a;X)} \to 0 \text{ uniformly in } f \text{ as } t_2-t_1 \to 0 \text{ by cond. 2.}$$

$\Rightarrow$ Equicontinuous.

$\forall t \in (0,T-a)$ $\{M_a f(t), F\}$ is pre-compact in X by 1 by Arzela-Ascoli, $M_a F$ is pre-compact in $C(0,T;X)$.

2.2 Error between local mean and pointwise values.
$$\| M_a f(t) - f(t)\| = \frac{1}{a}\int_t^{t+a} f(s) - f(t)\,ds.$$

$$= \frac{1}{a}\int_0^a f(t+s) - f(t)\, ds = \frac{1}{a}\int_0^a \tau_s f(t) - f(t)\, ds.$$

Thus, $\|M_a f - f\|^p_{L^p(0,T-a;X)}$

$$= \int_0^T \|M_a f(t) - f(t)\|_X\, dt$$

$$\leq \int_0^{T-a} \frac{1}{a}\left\|\int_0^a (\tau_a f - f)(t)\, ds\right\|_X^p dt$$

$$\leq \frac{1}{a^p}\int_0^{T-a}\left(\int_0^a \|(\tau_s f - f)t\|_X\, ds\right)^p dt$$

$$\leq \frac{1}{a^p}\int_0^{T-a}\int_0^a \left(\sup_s \|(\tau_s f - f)(t)\|_X\right)^p dt$$

$$= \int_0^{T-a} \sup \|(\tau_s f - f)(t)\|_X^p\, dt$$

$$= \sup \|\tau_s f - f\|_{L^p(T-a,X)}$$

It follows from (2), $\forall\, t_1 < T$

$$\|M_a f - f\|_{L^p(0,T,X)} \leq \|M_a f - f\|_{L^p(0,T-a,X)}$$

$$\leq \sup_{0\leq s\leq a} \|\tau_s f - f\|_{L^p(0,T-a,X)}$$

$\to 0$ as $a \to 0$ uniformly in F precompact in $L^p(0,T,X)$. by (2). equicontinuity.

$\boxtimes$

Feb. 25 - 25

$\nu Au + B(u,u) = f$, $f \in H$.

→ Galerkin, $\nu Au_m + P_m B(u_m, u_m) = P_m f$, (*)

$u_m \in H_m = \text{span}\{\varphi_1, \dots, \varphi_m\}$.

$\exists\ u_m \in H_m$ that solves (*). Also,
$\|u_m\| \le R_1 = \dfrac{|f|}{\nu \lambda_1^{1/2}}$. So $u_m \in V \underset{\text{comp.}}{\hookrightarrow} H$, so

$\exists\ u_{m_j} \underset{w}{\rightarrow} u$ in V, and $u_{m_j} \rightarrow w$ in H.

<u>Ex:</u> $w = u$; $\exists \{u_{m_j}\}_j$ s.t. $u_{m_j} \underset{w}{\rightarrow} u$ in V and

$u_{m_j} \underset{s}{\rightarrow} u$ in H strongly. Now fix $K \in \mathbb{N}$,

φ_K. $(\nu A u_m, \varphi_K)_H + (P_m B(u_m, u_m), \varphi_K) = (P_m f, \varphi_K)$.

Let $m \ge K$.

$\nu((u_m, \varphi_K)) + (B(u_m, u_m), \varphi_K) = (f, \varphi_K)$.

If $m_j \ge K$,

$\nu((u_{m_j}, \varphi_K)) + (B(u_{m_j}, u_{m_j}), \varphi_K) = (f, \varphi_K)$; $j \to \infty \Rightarrow$

$\nu((u, \varphi_K)) + \underbrace{\qquad\qquad\qquad}\ = (f, \varphi_K)$

$(B(u,\varphi,w) = \int (u\cdot\nabla) v\, w\, dx.$

$\langle B(u,v)w\rangle = -\langle B(u,w), v\rangle.$

Claim $(B(u_{m_j}, u_{m_j}), \varphi_k) \xrightarrow{j\to\infty} \langle B(u,u), \varphi_k\rangle_{V'\times V}.$

Proof: $|\langle B(u_{m_j}, u_{m_j}) - B(u,u), \varphi_k\rangle_{V'\times V}|$

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad ||$

$|\langle B(u_{m_j} - u, u_{m_j}) + B(u, u_{m_j} - u), \varphi_k\rangle| =$

$|\int (u_{m_j} - u)\cdot\nabla u_{m_j} w\, dx + \int (u\cdot\nabla)(u_{m_j} - u)\cdot\varphi_k\, dx|$

$\leq \int |u_{m_j} - u|\, |\nabla u_{m_j}|\, |w|\, dx + |\int (u\cdot\nabla)(u_{m_j} - u)\varphi_k\, dx|$

$\leq \|u_{m_j} - u\|_{L^3} \|\nabla u_{m_j}\|_{L^2} \|\varphi_k\|_{L^6} + |\int (u\cdot\nabla)\varphi_k (u_{m_j} - u) dx|$

$\leq \|u_{m_j} - u\|_{L^3} \|\nabla u_{m_j}\|_{L^2} \|\varphi_k\|_{L^6} + \|u\|_{L^6} \|\nabla\varphi_k\|_{L^2} \|u_{m_j} - u\|_{L^3}$

$\leq \underbrace{c\, |u_{m_j} - u|_{L^2}^{1/2}}_{\to 0} \|u_{m_j} - u\|^{1/2} \|u_{m_j}\|_{H^1} \|\varphi_k\|_{H^1} + C\|u\|\, \|\varphi_k\|\, \|u_{m_j} - u\|^{1/2} |u_{m_j} - u|^{1/2}$

SET + Interpolation $\quad \leq \sqrt{2} R_1^{1/2} \quad R_1 \quad$ fixed $\quad \leq R_1 \quad$ fixed $\quad \sqrt{2} R_1^{1/2} \quad 0$

$\longrightarrow 0.$ $\boxed{}$

$\Rightarrow \nu((u, \varphi_k)) + \langle B(u,u), \varphi_k\rangle_{V'\times V} = (f, \varphi_k)$

$\nu\langle Au, \varphi_k\rangle_{V'\times V} \quad\quad \langle f, \varphi_k\rangle_{V'\times V}$

$\quad\quad ||$

$\langle \nu Au + B(u,u), \varphi_k\rangle_{V'\times V} = \langle f, \varphi_k\rangle_{V'\times V} \quad \forall \varphi_k!$

$\Rightarrow \langle \nu A u + B(u,u) - f, \sum_{1}^{N} \underbrace{(w, \varphi_k)\varphi_k}_{\downarrow\, w} \rangle = 0 \quad \forall N.$

$\Rightarrow \langle \nu A u + B(u,u) - f, w \rangle_{V' \times V} = 0 \quad \forall w \in V.$

$\Rightarrow \nu A u + B(u,u) - f = 0$ in V'. $\rightarrow$ $\boxed{\text{Solution!}}$
$u \in V \subset H^1$

$\underset{\overset{\cap}{H^{-1}}}{-\nu \Delta u} + \underset{\overset{?}{H^{-1}}}{(u \cdot \nabla) u} - f \in H^{-1}$ (Claim)

$\langle u \cdot \nabla u, \varphi \rangle_{H^{-1} \times H^1} = \int (u \cdot \nabla u) \varphi \, dx \leq \|u\|_{L^3} \|\nabla u\|_{L^2} \|\varphi\|_{L^6}$

$\leq C \|\varphi\|_{H^1}$.

$\Rightarrow$ Bounded functional, in H^{-1}, so apply H-decomp

$\underbrace{-\nu \Delta u + (u \cdot \nabla) u}_{*} - f = \underbrace{\mathbb{P}_\sigma (\cdot *)}_{H^{-1}} - \underbrace{\nabla g}_{H^{-1}}$.

$= \underbrace{\nu A u + B(u,u) - f}_{0} - \nabla g$

$\Rightarrow -\nu \Delta u + (u \cdot \nabla) u + \nabla g = f, \; \mathrm{div}(u) = 0$, in $H^{-1} \subset \mathcal{D}'(\Omega)$.

$\hookrightarrow$ Existence, now want regularity so $u \in H^2$, $\mathrm{div}(u) = 0$.

<u>Regularity Result</u>: Let $f \in H$, then $u \in \mathcal{D}(A) = H^2 \cap V$.

1^{st} approach: Back to Galerkin

$$\nu A u_m + P_m B(u_m, u_m) = P_m f.$$

$\exists\ u_m \in H_m = \text{span}\{\underbrace{\varphi_1, \dots, \varphi_m}_{\text{eigen of } A}\} \subset D(A)$.

Can find $|Au_m| \le ???$

$$\nu |Au_m|^2 + (P_m B(u_m, u_m), \underset{Au_m}{\overset{\shortparallel}{Au_m}}) = (P_m f, Au_m)$$

$$\nu |Au_m|^2 = (-B(\underbrace{u_m}_{L^6}, \underbrace{u_m}_{L^3}), \underbrace{Au_m}_{L^2}) + (f, Au_m).$$

$$\nu |Au_m|^2 \le \|u_m\| \cdot \|u_m\|^{1/2} |Au_m|^{1/2} |Au_m| + |f| \, |Au_m|.$$

$$\le R_1^{3/2} |Au_m|^{3/2} + |f| \cdot |Au_m|$$

$$\nu |Au_m| \le R_1^{3/2} |Au_m|^{1/2} + |f| \quad , \quad xy \le \frac{x^2}{2} + \frac{y^2}{2}$$

$$\Rightarrow |Au_m| \le \frac{R^3}{2\nu^2} + \frac{|Au_m|}{2} + \frac{|f|}{\nu}$$

$$|Au_m| \le \frac{R_1^3}{\nu^2} + 2\frac{|f|}{\nu} := R_2.$$

So uniform bound in H^2, so get weakly convergent subsequence in H^2 and strongly convergent in H^1.

If $u \in H^2$, $u \cdot \nabla u \in L^2$.

Approach 2: $\exists\ u \in V$ s.t. $\nu A u + B(u,u) = f$ in V'.

Consider $A^{1/2}(P_m u)$

$$\langle \nu A u, \underset{\parallel}{A^{1/2} P_m u} \rangle + \langle B(u,u), A^{1/2} P_m u \rangle = \langle f, A^{1/2} P_m u \rangle.$$

$\nu |A^{3/4} P_m u|^2$

$\boxed{\text{Feb-27-25}}$ → Steady State Problem.

$$\nu Au + B(u,u) = f, \quad f \in H.$$

We have seen $\exists\, u \in V$ s.t. u is a solution, w/ $\|u\| \le R_1 = \dfrac{|f|}{\nu \lambda_1^{1/2}}$.

Claim $u \in \mathcal{D}(A) = H^2 \cap V$.

<u>Proof of approach 2:</u>

$$\nu Au + B(u,u) = f \text{ in } V', \; u \in V.$$

Act against a test function, $A^{1/2}(P_m u)$ ⟶ $P_m u \in V$, anywhere I want.

$$\langle \nu Au, A^{1/2}P_m u\rangle_{V' \times V} + \langle \underbrace{B(u,u)}_{L^{6/5}\, L^2}, \underbrace{A^{1/2}(P_m u)}_{L^3} \rangle_{V' \times V} = \langle f, A^{1/2}P_m u \rangle_{V' \times V}$$

$$\Rightarrow |\nu A^{3/4} P_m u|^2 \le C \|u\|_{L^6} \|\nabla u\|_{L^2} \|A^{1/2}P_m u\|_{L^3} + |f| \cdot \underbrace{|A^{1/2}P_m u|}_{\le R_1}.$$

Observe $u \in V$, $|A^{1/2}u| = \|u\|$, so

$$\|A^{1/2}P_m u\| = \|P_m u\| \le \|u\| \le R_1 \Rightarrow$$

$$\nu |A^{3/4}P_m u|^2 \le c \|u\| \|u\| \|A^{1/2}P_m u\|_{H^{1/2}} + |f| R_1.$$

$$\nu |A^{3/4}P_m u|^2 \le C R_1^2 \|A^{1/2}P_m u\|_{H^{1/2}} + |f| R_1.$$

$$\le C R_1^2 |A^{3/4}P_m u| + |f| R_1,$$

$$\le \frac{C^2 R_1^4}{2\varepsilon^2} + \frac{\varepsilon^2}{2} |A^{3/4}P_m u|^2 + |f| R_1, \quad \text{take } \frac{\varepsilon^2}{2} = \frac{\nu}{2}, \; \varepsilon = \sqrt{\nu}$$

$\frac{\nu^2}{2\varepsilon^2} \times \frac{\varepsilon^2 \nu}{\nu}$

ε makes units work out

$$\Rightarrow \frac{\nu}{2}|A^{3/4}P_m u|^2 \le \frac{C^2R_1^4}{2\nu} + |f|\cdot R_1. \Rightarrow$$

$$|A^{3/4}P_m u|^2 \le \frac{C^2R_1^2}{\nu^2} + \frac{2|f|R_1}{\nu} =: R_{3/2}^2$$

$$\sum_{K=1}^{m} \lambda_1^{3/2}|(u,\varphi_K)|^2 \le R_{3/2}^2 \Rightarrow \sum_{K=1}^{\infty} \lambda_1^{3/2}|(u,\varphi_K)|^2 \le R_{3/2}^2$$

$$\Rightarrow |A^{3/4}u|^2 \le R_{3/2}^2. \Rightarrow u \in D(A^{3/4}) \Rightarrow u \in H^{3/2} \cap V.$$

$\rightarrow$ New BootStrap: Repeat w/ $AP_m u$.

$$\langle \nu AP_m u, AP_m u\rangle + \langle B(\underset{L^6}{u},\underset{L^3}{u}), \underbrace{AP_m u}_{L^2}\rangle = \langle \underset{L^2}{f}, \underbrace{AP_m u}_{L^2}\rangle$$

$$\nu|AP_m u|^2 \le C\|u\|_{L^6}\|u\|_{L^3}\|AP_m u\|_{L^2} + |f|\,|AP_m u|$$

$$\le C\|u\|\,\underbrace{\|\nabla u\|_{H^{1/2}}}\,|AP_m u|_{L} + |f|\,|AP_m u|$$

This is legit b.c. of last argument!

$$\le CR_1R_{3/2}|AP_m u| + |f|\,|AP_m u| \text{ ; divide:}$$

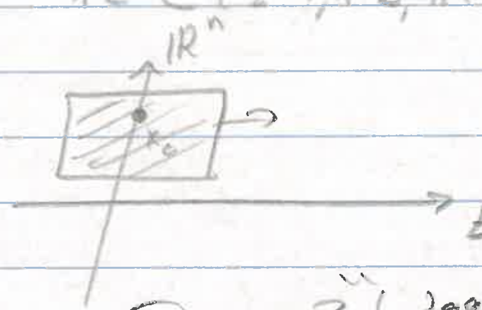
$$\Rightarrow |AP_m u| \le \frac{CR_1R_{3/2}}{\nu} + \frac{|f|}{\nu}. \Rightarrow |Au| \le \frac{CR_1R_{3/2}}{\nu} + \frac{|f|}{\nu}$$

$\Rightarrow u \in D(A)$. ∎

$\boxed{\rightarrow \text{ ALWAYS HOLD TIGHT TO SIMPLE IDEAS}}$

→ want C^1

$$\begin{cases} \dot{x} = F(x,t), \\ x(0) = x_0 \end{cases} \quad \exists! \text{ of } x \in C^1([-T,T], \mathbb{R}^n).$$



"Weak form" of ODE problem.

$$x(t) = \int_0^t F(x(\tau),\tau)\,d\tau + x_0.$$

→ First find an x that is continuous only!

$$x \in C^0([-T,T], \mathbb{R}^n).$$

$\exists!$ $x \in C^0$, but since x solves the equation, we get $x \in C^1$! Same Philosophy, solve "easier" problem, then bring it to desired space by using its being a solution! Same idea always in PDEs.

Back to stationary N-S:

Q: Find conditions under which
$\nu A u + B(u,u) = f$ has unique solution,
w/ $f \in H$, $u \in D(A)$

→ If $f=0$, $\nu(Au,u)$, $\nu\|u\|^2 + \underbrace{(B(u,u),u)}_{0} = 0$ $\Longrightarrow$

$u = 0$.

Solution: Let u_1, u_2 be 2 solutions,

$w = u_1 - u_2$, $\|u_j\| \le R_1$, $j = 1, 2$
$\overset{\in}{D(A)}$

$$\nu A w + B(w, u_2) + B(u_1, w) = 0 \text{ in } H.$$

$$\nu |Aw|^2 + (B(w, u_2), w) + \underbrace{(B(u_1, w), w)}_{0} = 0.$$

$$\nu |Aw|^2 + (B(w, u_2), w) = 0. \Rightarrow$$

$$\nu |Aw|^2 = -(B(w, u_2), w), \quad D^2 \le C D^2 \Rightarrow$$

$(1-c) D^2 \le 0$,

so $c < 1 \Rightarrow D = 0$.

$$\nu \|w\|^2 = |(B(w, u_2), w)| \le \|w\| \cdot \|u_2\| \, |w|^{1/2} \|w\|^{1/2}$$
$$\le \|w\|^{3/2} R_1 |w|^{1/2}$$
$$\le \|w\|^{3/2} R_1 \frac{\|w\|^{1/2}}{\lambda_1^{1/4}}$$

$\Rightarrow \left(\nu - \frac{R_1}{\lambda_1^{1/4}}\right) \|w\|^2 \le 0$, if $\nu > \frac{R_1}{\lambda_1^{1/4}}$ we

have uniqueness, $\boxed{\frac{|f|}{\nu^2 \lambda_1^{3/4}} < 1}$

$\longrightarrow G$, Grashoff #

alternative to Reynolds number.

Mar-18-25

$$\begin{cases} \partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f, \\ \operatorname{div}(u) = 0 \\ u(0) = u_0(x) \end{cases}$$

$$\begin{cases} \frac{du}{dt} + \nu A u + B(u,u) = \mathbb{P}_\sigma(f) (= f) \\ f, u_0 \in H \end{cases}$$

→ Let us focus on linear:

$$\begin{cases} \frac{du}{dt} + \nu A u = f \\ u(0) = u_0(x) \end{cases}$$

assume $f \in H$, $u_0 \in H$

Formally, $u = \sum_{k=1}^{\infty} a_k(t) \varphi_k(x)$, we get

$$\sum_{k=1}^{\infty} \dot{a}_k(t) \varphi_k(x) + \nu \sum_{k=1}^{\infty} a_k(t) \lambda_k \varphi_k(x) = \sum_{k=1}^{\infty} f_k \varphi_k$$

$$\underbrace{u_0(x) = \sum a_k(0) \varphi_k(x) ; \quad a_k(0) = (u_0, \varphi_k) \quad f_k = (f, \varphi_k)}_{\text{Given}}$$

→ ip w/ φ_j ⇒

$$\dot{a}_j(t) + \nu \lambda_j a_j(t) = f_j, \quad \forall j, \quad a_j(0) = (u_0, \varphi_j)$$

→ use this info rigorously.

$$\Rightarrow \quad a_j(t) = e^{-\nu\lambda_j t} a_j(0) + f_j \frac{(1-e^{-\nu\lambda_j t})}{\nu\lambda_j}.$$

$u_m := \sum_{j=1}^{m} a_j(t)\varphi_j(x) \in H_m$ → Very smooth

↳ See if these converge!

function in time

→ Look at squares... $|u_m - u_n|^2 = |\sum_{j=n}^{m} a_j\varphi_j|^2 = \sum_{j=n}^{m} |a_j|^2$.

$$\frac{d}{dt}|a_j|^2 + \nu\lambda_j |a_j|^2 = f_j a_j \le |f_j|\cdot|a_j| \le \frac{|f_j|^2}{2\nu\lambda_j} + \frac{\nu\lambda_j}{2}|a_j|^2$$

$$\Rightarrow \quad \frac{d}{dt}|a_j|^2 + \nu\lambda_j |a_j|^2 \le \frac{|f_j|^2}{\nu\lambda_j} \quad \forall t \in \mathbb{R}$$

↳ $* \; e^{\nu\lambda_j t} > 0$

$$\frac{d}{dt}\left(|a_j(t)|^2 e^{\nu\lambda_j t}\right) \le \underbrace{\frac{|f_j|^2}{\nu\lambda_j} e^{\nu\lambda_j t}}_{\text{integrate}}, \text{ for } t \ge 0.$$

$$|a_j(t)|^2 \le e^{-\nu\lambda_j t}|a_j(0)|^2 + \frac{|f_j|^2}{(\nu\lambda_j)^2}(1-e^{-\nu\lambda_j t})$$

$\lambda_j \to \infty$

$$\Rightarrow \sum_n^m |a_j(t)|^2 \le \sum_n^m |a_j(0)|^2 e^{-\nu\lambda_j t} + \sum_n^m \frac{|f_j|^2}{(\nu\lambda_j)^2}$$

$$\le e^{-\nu\lambda_1 t}\sum_n^m |a_j(0)|^2 + \frac{1}{(\nu\lambda_1)^2}\sum_n^m |f_j|^2$$

For $t \in [0,T]$ $\to$ Cauchy.

$$\sup_{0 \le t \le T} |u_m(t) - u_n(t)|^2 \le \sum |(u_0, \varphi_k)|^2 + \frac{1}{\nu\lambda_1} \sum_{k=n}^{m} |(f, \varphi_k)|^2$$

$\Rightarrow$ $u_m \to u$ in $C([0,T], H)$.

$\|u_m\|_V$? $\to$ (cla V-norm)

$\|u_m - u_n\|^2 \le$?

Repeat, but achieve:

$$\lambda_j |a_j(t)|^2 \le \lambda_j e^{-\nu\lambda_j t} |a_j(0)|^2 + \frac{|f_j|^2}{(\nu\lambda_j)^2} (1 - e^{-\nu\lambda_j t}) \lambda_j .$$

$\boxed{\to 0}$ $\Rightarrow$ converge in V.

$$\Rightarrow \sum_n^m \lambda_j |a_j(t)|^2 \le \sum_n^m \lambda_j e^{-\nu\lambda_j t} |a_j(0)|^2 + \sum_n^m \frac{1}{\nu^2 \lambda_j} |f_j|^2$$

For max forma

Look at: $\sup_{j=1,2,\dots} (\lambda_j e^{-\nu\lambda_j t}) \le \sup_{x \ge \lambda_1} (x e^{-\nu x t})$

for $[\lambda_1, \infty)$.

$f(x) = x e^{-\nu x t}$, $\lim_{x \to \infty} f(x) = 0$, $f(\lambda_1) = \lambda_1 e^{-\nu\lambda_1 t}$.

$f'(x) = -\nu t x e^{-\nu x t} + e^{-\nu x t} = 0 \Rightarrow$ $\boxed{x = \frac{1}{\nu t}} =: x^*$.

when is $\lambda_1 \le x^*$ or $x^* < \lambda_1$?

$f(x^*) = \frac{1}{\nu t} e^{-1}$, so $\sup f(x) = m(\nu, \lambda_1, t)$

$$= \max\left\{ \frac{e^{-1}}{\nu t}, \lambda_1 e^{-\nu\lambda_1 t} \right\}.$$

In summary: $\forall \tau > 0$ $C([\tau, T], V)$, $u_m \to u$

Can show for $t>0$ $\|u(t)\|^2 \le m(\nu, \lambda_1, t)|u_0|^2 + \frac{1}{\nu^2\lambda_1}|f|^2$.

$\to$ ~~since $m \sim \frac{1}{t}$, not integrable from this analysis. Be more careful.~~

$$\int_0^T \|u_m(t) - u_n(t)\|^2 \, dt \le \sum_n^m \frac{1}{\nu}\big(\underbrace{1 - e^{-\nu\lambda_j T}}_{\le 1}\big)|g_j(0)|^2 + \underbrace{\left(\sum_n^m \frac{1}{\nu\lambda_1}|f_j|^2\right) T}_{\text{Cauchy}}$$

So convgs in $L^2((0,T), V)$, so $u_m \to u$ in

$\frac{du}{dt}$, νAu in $L^2((0,T), V')$, f $\to$ converging in $L^2((0,T), V')$

<u>Summary:</u> $\frac{d}{dt}u_m + \nu A u_m = P_m f$ (H_m)

$u_m \to u$ in $C([0,T], H)$, $C([\tau,T], V)$ $\forall \tau > 0$,
$L^2((0,T), V)$, $\frac{du}{dt} \in L^2((0,T), V')$.

$\to$ $Au_m \to Au$ $L^2((0,T), V')$.

Mar-20-2025

→ Last time: $\partial_t u - \nu \Delta u + \nabla p = f$

$\operatorname{div}(u) = 0$

$u(x,0)$

$f \in L^2 \Rightarrow f \in H, \; u_0 \in H.$

$$\begin{cases} \dfrac{du}{dt} + \nu A u = f \\ u(0) = u_0 \end{cases}, \quad u_m = \sum_{1}^{m} a_k \varphi_k$$

$$\begin{cases} \dot{a}_k + \nu \lambda_k a_k = f_k \\ a_k(0) = (u_0, \varphi_k) \end{cases}$$

u_m is Cauchy in $C([0,T], H)$ $\forall T > 0$,

$C([\tau,T], V)$ $\tau > 0$

Au_m is Cauchy in $L^2((0,T), V)$

$L^2((0,T), V')$

$\Rightarrow \dfrac{du_m}{dt}$ Cauchy in $L^2((0,T), V')$

$\Rightarrow$ $u_m \to u$ in relevant norms.

$$\Rightarrow \begin{cases} \dfrac{du}{dt} + \nu A u = f \text{ in } L^2((0,T), V'), \\ u(0) = u_0 \end{cases}$$

$\dfrac{du}{dt} + \nu A u - f \in V'$ for a.e. $t \in [0,T]$

↳ Distribution $\Rightarrow \exists\, \mu$ s.t.

$\mu = \dfrac{du}{dt} + \nu A u - f$

Let $\varphi \in \mathcal{V}$ → div-free test function.

$$\left\langle \frac{du}{dt}, \varphi \right\rangle_{V'} + \nu ((u, \varphi)) - (f, \varphi) = 0 \text{ in } L^2([0,T])$$

$\to$ dn distribution

$\to$ Rewrite $\langle \frac{du}{dt}, \varphi \rangle - \nu\langle \Delta u, \varphi\rangle - \langle f, \varphi\rangle = 0$

$\langle \frac{du}{dt} - \nu\Delta u - f, \varphi\rangle = 0$ in $L^2([0,T])$ a.e.

By De Rham, $\exists\ g \in \mathcal{D}'(\Omega)$ s.t.

$\frac{du}{dt} - \nu\Delta u - f = -\nabla g.$

Q: Does solution depend cont. on initial data?

$u(t)$, $u(0)=u_0$, $v(t)$, $v(0)=v_0$.

$w = u - v \Rightarrow w(0) = u_0 - v_0 \Rightarrow$

$$\begin{cases} \frac{dw}{dt} + \nu A w = 0 & \text{holds in } L^2([0,T], V') \\ w(0) = w_0 & w \in L^2((0,T), V) \end{cases}$$

use this property

Nonlinear Case does not enjoy this property!

$\langle \frac{dw}{dt}, w\rangle_{V'\times V} + \underbrace{\nu(Aw, w)_{V'\times V}}_{\nu\|w\|^2} = 0$. Apply Theorem $\Rightarrow$

$e^{2\nu\lambda_1 t}$

$\frac{d}{dt}\frac{|w|^2}{2} + \underbrace{\nu\|w\|^2}_{\geq \nu\lambda_1|w|^2} = 0 \Rightarrow \frac{d}{dt}|w|^2 + 2\nu\lambda_1|w|^2 \leq 0.$

$\frac{d}{dt}\left(e^{2\nu\lambda_1 t} w^2\right) \leq 0 \quad |w(t)|^2 \leq e^{-2\nu\lambda_1 t}|w(0)|^2.$

$\Rightarrow \sup_{0\leq t\leq T}|w(t)|^2 \leq |w(0)|^2.$

$H \to L^\infty([0,T), H)$

$\Phi(u_0) \to u(t) \Rightarrow \|\Phi(u_0) - \Phi(v_0)\|_{L^\infty} \le |u_0 - v_0|$

$\Rightarrow$ Φ is Lipschitz.

Nonlinear!

3D

$$\begin{cases} \partial_t u + \nu \Delta u + (u\cdot\nabla)u + \nabla p = f \\ \operatorname{div}(u) = 0 \\ u(x,0) = u_0(x) \end{cases}$$

$\to$ Solve this functional equation

$$\begin{cases} \frac{du}{dt} + \nu A u + B(u,u) = f \\ u(0) = u_0 \end{cases}$$

Formal

Look at Galerkin, $u = \sum_1^\infty \hat{a}_k \varphi_k \Rightarrow$

$$\sum_1^\infty \dot{\hat{a}}_k \varphi_k + \sum_{k=1}^\infty \nu \lambda_k \hat{a}_k \varphi_k + B\left(\sum_1^\infty \hat{a}_\ell \varphi_k, \sum_1^\infty \hat{a}_m \varphi_m\right) = \sum_1^\infty f_k \varphi_k$$

ip w/ φ_j

$$\sum_1^\infty \dot{\hat{a}}_k \varphi_k + \sum_1^\infty \nu \lambda_k \hat{a}_k \varphi_k + \sum_1^\infty \sum_1^\infty \hat{a}_\ell \hat{a}_m \, B(\varphi_\ell, \varphi_m) = \sum_1^\infty f_k \varphi_k$$

$\to j = 1, \dots,$

$$\dot{\hat{a}}_j + \nu \lambda_j \hat{a}_j + \sum_{m=1}^\infty \sum_{\ell=1}^\infty \hat{a}_\ell \hat{a}_m (B(\varphi_\ell, \varphi_m), \varphi_j) = f_j$$

$\to$ Truncate, $j = 1, \dots, N$.

$\xrightarrow{} j = 1, \ldots, N$

$$\dot{a}_j - \nu \lambda_j a_j + \sum_{l,m=1}^{N} a_l a_m \underbrace{(B(\varphi_l, \varphi_m), \varphi_j)}_{c_{lmj} \to \text{known}} = f_j$$

$a_j(0) = (u_0, \varphi_j) \to$ Given

→ Since this is finite system of ODEs quadratic, we apply Picard to get a unique solution for some time interval $(-\tau_N, \tau_N)$.

$$\to \begin{cases} \dfrac{du_N}{dt} + \nu A u_N + P_N B(u_N, u_N) = P_N f \\ u_N(0) = P_N u_0 \end{cases}$$

→ That u_N is bounded, let $\tau_N^* =$ be maximal interval of existence of $t > 0$.

↳ $[0, T_N^*)$

If $T_N^* = \infty$, $\Downarrow$ $T_N^* < \infty$, then

$$\limsup_{t \to (T_N^*)^-} \|u_N\| = \infty$$

Take ip w/ u_N ⇒

$$\frac{1}{2} \frac{d}{dt} |u_N|^2 + \nu \|u_N\|^2 + \underbrace{(P_N B(u_N, u_N), u_N)}_{0} = (P_N f, u_N).$$

$$\Rightarrow \quad \frac{1}{2} \frac{d}{dt} |u_N|^2 + \nu \|u_N\|^2 \leq |f| \, |u_N| \leq |f| \, \frac{\|u_N\|}{\lambda_1^{1/2}} \leq \frac{|f|^2}{2\nu\lambda_1} + \frac{\nu}{2} \|u_N\|^2$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}|u_N|^2 + \frac{\nu}{2}\|u_N\|^2 \le \frac{|f|^2}{2\lambda_1^{1/2}} \Rightarrow$$

$$\frac{d}{dt}|u_N|^2 + \nu\|u_N\|^2 \le \frac{|f|^2}{\nu\lambda_1^{1/2}} \Rightarrow$$

$$\frac{d}{dt}|u_N|^2 + \nu\lambda_1|u_N|^2 \le \frac{|f|^2}{\nu\lambda_1^{1/2}} \quad // \times e^{\nu\lambda_1 t} \Rightarrow$$

$$|u_N(t)|^2 \le e^{-\nu\lambda_1 t}\underbrace{|u_N(0)|^2}_{\substack{\shortparallel \\ P_N u_0 \\ \text{IN} \\ |u_0|^2}} + \frac{|f|^2}{(\nu\lambda_1)^2}(1-e^{-\nu\lambda_1 t}).$$

$$\Rightarrow |u_N(t)|^2 \le e^{-\nu\lambda_1 t}|u_0|^2 + \frac{|f|^2}{(\nu\lambda_1)^2}(1-e^{-\nu\lambda t})$$

$$\Rightarrow |u_N(t)|^2 \le |u_0|^2 + \frac{|f|^2}{(\nu\lambda_1)^2} \rightarrow$$ independent of N!

$\Rightarrow$ limsup is bounded, so we have infinite existence forward in time.

$$\Rightarrow \|u_N\|_{L^\infty((0,T),H)} \le \text{Bnd.}$$

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Weak Solutions:

$$\begin{cases} \partial_t u + \nu A u + B(u,u) = f, \quad u(0) = u_0 \\ u_0, f \in H \text{ given} \end{cases}$$

Galerkin ODE system:

$$\frac{du_m}{dt} + \nu A u_m + P_m B(u_m, u_m) = P_m f,$$

$\exists !$ solution $[0,T]$, $\forall T > 0$,

$$\frac{d}{dt} |u_m(t)|^2 + \nu \lambda_1 \|u_m\|^2 \leq \frac{|f|^2}{\nu \lambda_1} \Rightarrow$$

$$|u_m(t)|^2 \leq e^{-\nu \lambda_1 t} |u_0|^2 + \frac{|f|^2}{(\nu \lambda_1)^2} (1 - e^{-\nu \lambda_1 t})$$

$$\leq K_0^2, \quad K_0 = K_0(\nu, \lambda_1, u_0, u_0) = |u_0|^2 + \frac{|f|^2}{(\nu \lambda_1)^2}.$$

$\Rightarrow \|u_m\|_{L^\infty((0,T)H)} \leq K_0$, by Banach-Alaoglu,

$\exists\, u_{m_j}$ s.t. $u_{m_j} \xrightarrow{w-*} u$ in $L^\infty((0,T), H)$.

$$|u_m(t)|^2 + \int_0^t \|u_m(\tau)\|^2 d\tau \leq T \frac{|f|^2}{(\nu \lambda_1)} + \underbrace{|u_m(0)|^2}_{\leq |u_0|^2}$$

$$\Rightarrow \int_0^T \|u_m(\tau)\|^2 d\tau \leq \frac{|f|^2}{\nu \lambda_1} T + |u_0|^2.$$

$$\Rightarrow \int_0^T \|u_m(\tau)\|^2 d\tau \leq \overline{K}_1^2 = \overline{K}_1^2 (T, f, u_0, \lambda_1).$$

$$\Rightarrow \|u_m\|_{L^2(0,T,V)} \le \overline{K_1}.$$ → independent of m, but not on time.

$$\Rightarrow u_{m_j} \xrightarrow{w} u \text{ in } L^2(0,T,V).$$

→ Back to equation

$$\left(\frac{d}{dt}u_m, \varphi_k\right) + \nu((u_m,\varphi_k)) + (P_m B(u_m,u_m),\varphi_k) = (P_m f, \varphi_k)$$

$$\Rightarrow (u_{m_j}(t),\varphi_k) + \nu\int_0^t ((u_{m_j}(\tau),\varphi_k))d\tau + \int_0^t (B(u_{m_j},u_{m_j})\varphi_k)$$
$$= (u_0,\varphi_k) + (f,\varphi_k)t$$

Since $u_{m_j} \xrightarrow{w*} u \Rightarrow$ ↗ What to do with this?

$$(u(t),\varphi_k) + \nu\int_0^t ((u(\tau),\varphi_k))d\tau + \lim_{m_j\to\infty}\int_0^t (B(u_{m_j},u_{m_j}),\varphi_k)$$
$$= (u_0,\varphi_k) + (f,\varphi_k)t.$$

$$\int_0^T \|u_m(\tau)\|^2 d\tau \le K_1^2 \Longleftrightarrow \int_0^T \|Au_m(\tau)\|^2_{V'} \le \overline{K_1}^2$$

$$\|Au_m\|_{L^2(0,T),V')} \le \overline{K_{II}} \rightarrow (3P)$$

→ Can obtain: $\|P_m B(u_m,u_m)\|_{V'} \le |u_m|^{1/2}\|u_m\|^{3/2} \in L^{4/3}(0,T).$

$$\le K_0^{1/2}$$

$$\Rightarrow P_m B(u_m,u_m) \in L^{4/3}(0,T,V')$$

Thus, $\left\| \frac{du_m}{dt} \right\|_{L^{4/3}(0,T,V')} \leq K'(T, \nu, \lambda_1, f)$.

To apply compactness theorem: $V \underset{comp}{\hookrightarrow} H \underset{cont}{\hookrightarrow} V'$

$p_1 = 2$, $p_0 = \infty$, $p_2 = 4/3$

$u_m \in L^{p_1}((0,T), V)$, $u_m \in L^{p_0}((0,T), H)$, $\frac{du_m}{dt} \in L^{p_2}(0,T,V)$,

$\Rightarrow \boxed{u_{m_j} \xrightarrow{strong} u \text{ in } L^2(0,T,H).}$

also, $\frac{du_m}{dt} \rightarrow \frac{du}{dt}$ in $L^{4/3}(0,T,V')$.

Because $u_{m_j} \to u$ in L^2, then $u_{m_j}(t) \to u(t)$ in H for a.e. $t \in [0,T]$.

$$\left| \int_0^t \langle B(u_m, u_m) - B(u,u), \varphi_k \rangle d\tau \right| =$$

$$\left| \int_0^t \langle B(u_m - u, u_m) + B(u, u_m - u), \varphi_k \rangle d\tau \right|$$

$$\leq \int_0^T |\langle B(\underbrace{u_m - u}_{L^3}, \underbrace{u_m}_{L^2}), \underbrace{\varphi_k}_{L^0} \rangle| + |\langle B(u, u_m - u), \varphi_k \rangle| \, d\tau$$

$$\leq \int_0^T |u_m - u|^{1/2} \underbrace{\|u_m - u\|^{1/2}}_{\leq (\|u_m\|^{1/2} + \|u\|^{1/2})\ (?)} (\|u_m\| \, \|\varphi_k\| + |u| \, \|\varphi_k\|) \, d\tau$$

$$\left(\int_0^t |u_m - u|^{1/2} \|u_m\|^{3/2} d\tau \right) \|\varphi_k\| + \frac{\to 0}{+} \to 0$$

4, $4/3$

$$\Big(\underbrace{\int_0^T |u_m - u|^2}_{\downarrow m \to \infty \ 0} \Big)^{1/4} \Big(\underbrace{\int_0^T \|u_m\|^2}_{\leq K_2^{3/4}} \Big)^{3/4} \|\varphi_k\| \qquad \int \underbrace{|u_m - u|^{1/2}}_{\substack{4 \\ \|u_m - u\|^2}} \underbrace{|u_m|^{1/2}}_{\substack{4 \perp \\ \|u_m\|^2}} \underbrace{\|u\|}_{2} \cdot \|\varphi_k\|$$

linear in φ_k.

$\Rightarrow$

$$(u(t),\varphi_k) - (u_0,\varphi_k) + \int_0^t ((u(\tau),\varphi_k))\,d\tau + \int_0^t \langle B(u,u),\varphi_k\rangle_{V',V}\,d\tau = (f,\varphi_k)t$$

$\rightarrow$ True for linear combos of φ_k, $\varphi = \sum_{j=1}^{N} \alpha_j \varphi_j$.

Claim: True for $\varphi \in V$, $\forall \varphi \in V$, $\forall \varepsilon > 0$, $\exists \varphi_\varepsilon$ finite combo s.t. $\|\varphi - \varphi_\varepsilon\|_V < \varepsilon$.

$$(u(t),\varphi-\varphi_\varepsilon) \le \|u(t)\| \cdot \|\varphi-\varphi_\varepsilon\| \le |u(t)|\,\|\varphi-\varphi_\varepsilon\|$$

$$\le K_0\,\|\varphi-\varphi_\varepsilon\| < K_0\,\varepsilon.$$

careful w/ $u(t)$? a.e. t

$$\left|\int ((u(\tau),\varphi-\varphi_\varepsilon))\,d\tau\right| \le \int_0^T \|u(\tau)\|\,\|\varphi-\varphi_\varepsilon\|$$

$$\le T^{1/2}\bar{K}_1 \cdot \varepsilon$$

$$\int_0^t |\langle B(u,u),\varphi-\varphi_\varepsilon\rangle|\,dt = \int_0^t \left|\int (u\cdot\nabla)u\,(\varphi-\varphi_\varepsilon)\,dx\right|d\tau$$

$$\le \int_0^t \int_D |\underbrace{u}_{L^4}\cdot\underbrace{\nabla u}_{L^2}\;\underbrace{(\varphi-\varphi_\varepsilon)}_{L^4}|\,dx\,d\tau$$

$$\le \int_0^t |u(\tau)|^{1/2}\,\|u(\tau)\|^{1/2}\,\|u(\tau)\|\,\|\varphi-\varphi_\varepsilon\|\,d\tau$$

$$\le K_0\,\varepsilon\,K_1^{3/2}\,T^{1/4} \to 0.$$

Mar-27-2025 $\varphi \in V$

$\langle u_{m_j}(t) - u_{m_j}(0), \varphi \rangle_{V' \times V} + \nu \int_0^t ((u_{m_j}(\tau), \varphi)) d\tau + \int \langle P_{m_j}(B(u_{m_j}, u_{m_j})), \varphi \rangle =$

$\int_0^t (g, \varphi) d\tau,$

since $u_{m_j} \to u$ in $C([0,T], V')$, $\langle u(t), \varphi \rangle - \langle u_0, \varphi \rangle +$

$\nu \int_0^t ((u(\tau), \varphi)) d\tau + \int_0^t \langle B(u,u), \varphi \rangle_{V' \times V} = (g, \varphi) t.$

$$\Rightarrow \quad u(t) - u_0 = \int_0^t \underbrace{-Au - B(u,u) + f}_{\in L^{4/3} \subset L^1} d\tau \quad \text{in } V'.$$

$\longrightarrow$ $u, g \in L^1((0,T), X)$,

$u'(t) + \nu Au + B(u,u) = f$ in V' and $\to L^{4/3}((0,T), V')$ $\longleftarrow$ $u(t) = \xi + \int_0^t g(s) ds \iff u'(t) = g(t)$

w/ Properties of u, ① $u \in L^\infty((0,T), H)$, $\|u\|_{L^\infty(H)} \le K_0$

② $u \in L^2((0,T), V)$ — can show that $|u(t)| \le K_0$ for all $t \in [0,T]$.

③ $\frac{du}{dt} \in L^{4/3}((0,T), V')$

④ $u \in C([0,T], V')$, ⑤ $u \in C([0,T], H_{weak})$.

$\rightsquigarrow$ We are unable to show uniqueness.

Energy inequality for Galerkin method solution:

$$\frac{|u(t)|^2}{2} - \frac{|u(t_0)|^2}{2} + \nu \int_{t_0}^t \|u(\tau)\|^2 d\tau \le \int_{t_0}^t (g, u) d\tau.$$

For every $t > 0$ and a.e. $t_0 < t$ ($t_0 = 0$ included) holds:

$$\frac{1}{2} |u_m(t)|^2 - \frac{1}{2} |u_m(t_0)|^2 + \nu \int_{t_0}^t \|u_m(\tau)\|^2 d\tau = \int_{t_0}^t (g, u_m) d\tau.$$

Equality b.c. FD, ODE system.

$\exists\, E \subset [0,T]$ s.t. $m(E^c) = 0$, and $\forall t \in E, s \in E$,
$|u_m(t)|^2 \to |u(t)|^2$, $|u_m(s)|^2 \to |u(s)|^2$,
$u_m \to u$ in $L^2((0,T), H)$

Then, for $t, t_0 \in E$:

$$\frac{1}{2}|u(t)|^2 - \frac{1}{2}|u(t_0)|^2 + \int_{t_0}^{t} \|u(\tau)\|^2 d\tau \le \int_{t_0}^{t} (f,u)\, d\tau$$

since $u_m \to u$ weakly in $L^2((0,T), V)$,

$$\int_{t_0}^{t} \|u(\tau)\|^2 d\tau \le \liminf \int_{t_0}^{t} \|u_m(\tau)\|^2 d\tau$$

To get every t: $t_0 \in E$, $t \in E^c$, $t > t_0$, $\exists\, t_k \subset E$ s.t. $t_k \to t$,

$$\frac{1}{2}|u(t_k)|^2 - \frac{1}{2}|u(t_0)|^2 + 2\nu \int_{t_0}^{t_k} \|u\|^2 d\tau \le \int_{t_0}^{t_k} (f,u)\, d\tau.$$

$$\frac{1}{2}|u(t)| - \frac{1}{2}|u(t_0)|^2 + \int_{t_0}^{t} \|u\|^2 d\tau \le \int_{t_0}^{t} (f,u)\, d\tau.$$

$|u(t_k)| \le K_0$, $\exists\, u(t_{k_j}) \to u(t)$ weakly, $\Rightarrow |u(t)| \le \liminf |u(t_{k_j})|$.

Solutions w/ these properties are called Leray-Hopf solutions.

apr-1-2025

$\to$ 3D NS, t-dependent

$$\begin{cases} \frac{d}{dt}u + \nu Au + B(u,u) = f & \text{in } L^{4/3}((0,T),V') \\ u(0) = u_0 \end{cases}$$

$u_0, f \in H$.

$\exists$ weak solution u: $u \in L^\infty((0,T),H) \cap L^2((0,T),V) \cap C([0,T],H_{weak})$
$\cap\, C([0,T],V')$, $\frac{du}{dt} \in L^{4/3}((0,T),V')$

Galerkin method solution also satisfies

$$\frac{d}{dt}\frac{|u(t)|^2}{2} + \frac{1}{2}|u(s)|^2 + \nu\int_s^t \|u(\tau)\|^2 d\tau \le \int_s^t (f, u(\tau))\,d\tau$$

$\forall t \in [0,T]$, a.e. s, $s=0$ included.

Leray-Hopf Solution.

2D NSE Weak Solutions $\to$ Problem is $\frac{d}{dt}$ of $L^{4/3}$, everything else is the same.

$$\frac{du_m}{dt} = -\nu Au_m - P_m B(u_m, u_m) + P_m f \implies$$

$$\underbrace{\left\|\frac{du_m}{dt}\right\|_{V'}}_{L^2((0,T))} \le \nu \underbrace{\|Au_m\|_{V'}}_{= \nu\|u_m\|} + \underbrace{\|P_m B(u_m,u_m)\|_{V'}}_{\text{Problem here}} + \underbrace{\|P_m f\|}_{L^2}$$

$$\|B(u_m,u_m)\|_{V'} = \sup_{\substack{\varphi\in V \\ \|\varphi\|=1}} |\langle B(u_m,u_m),\varphi\rangle_{V'}|$$

$$= \sup_\varphi \left|\int (u_m\cdot\nabla)u_m \varphi\,dx\right| = \sup_\varphi \left|\int (u_m\cdot\nabla)\varphi u_m\,dx\right|$$

$$\le \sup_\varphi \int |\underbrace{(u_m}_{L^4}\cdot\underbrace{\nabla)\varphi}_{L^2}\underbrace{u_m}_{L^4}|\,dx \le \underset{\|\nabla\varphi\|=1}{\|u_m\|_{L^4}^2}$$

$\xrightarrow{}$ SET

In 2D: $\|u\|_{L^4} \le C\|u\|_{H^{1/2}} \Rightarrow$

$\boxed{\begin{matrix} V = H^1 + \text{stuff} \\ H = L^2 + \text{stuff} \end{matrix}}$

$\le C\|u_m\|^2_{H^{1/2}}$

$\le C\|u_m\|_{L^2}\|u_m\|_{H^1} \in L^2(0,T)$

$\Rightarrow \quad \|B(u_m,u_m)\|_{V'} \in L^2(0,T)$, and so

$$\boxed{\frac{du}{dt} + \nu Au + B(u,u) = f \text{ in } L^2(0,T,V').}$$

$\hookrightarrow$ Not $L^{4/3}$! Sin $u \in L^2(0,T,V)$, I can take action on u, push forward!

$\Rightarrow \langle \frac{du}{dt}, u\rangle_{V'} + \nu\langle Au, u\rangle_{V'} + \langle B(u,u), u\rangle_{V'} = (f,u)_{V'}$

$\downarrow$

$\frac{1}{2}\frac{d}{dt}|u|^2 + \nu\|u\|^2 + 0 = (f,u)$. $\to$ Integrate $0 \le s \le t \le T$ $\forall s,t$.

$\frac{1}{2}|u(t)|^2 - \frac{1}{2}|u(s)|^2 + \nu\int_s^t \|u(\tau)\|^2 d\tau = \int_s^t (f,u)\,d\tau$,

$\hookrightarrow$ Energy Equality!

$\Rightarrow \frac{1}{2}\left||u(t)|^2 - \frac{1}{2}|u(s)|^2\right| \le \underbrace{\left|\int_s^t \|u(\tau)\|^2 d\tau\right| + \left|\int_s^t (f,u)\,d\tau\right|}_{\text{continuous, } s\to t \Rightarrow \to 0,}$ $\hookrightarrow f \in L^1 \Rightarrow \int f\,dx$ is cont.

and so $|u(t)|$ is continuous! $|u(t)| \to |u(t_0)|$ as $t \to t_0$, $u(t) \xrightarrow{w} u(t_0)$ in H.

Obs: Let $\{\varphi_n\} \subset H$ (Hilbert) s.t. $\varphi_n \to \varphi_0$ weakly in H and $|\varphi_n| \to |\varphi_0|$. Then $|\varphi_n - \varphi_0| \to 0$. $\quad |\varphi_n - \varphi_0|^2 = (\varphi_n - \varphi_0, \varphi_n - \varphi_0)$

$= |\varphi_n|^2 - 2(\varphi_n, \varphi_0) + |\varphi_0|^2 \to \boxed{0!}$

$\to |\varphi_0|^2 \quad 2(\varphi_0,\varphi_0) + |\varphi_0|^2$

$\blacksquare$

So, in 2D, $u \in C([0,T], H)$. (From energy estimate, which came from SET allowing us to test equation w/ u.)

THIS IS BIG DIFFERENCE

We can now prove uniqueness for 2D problem:

$w = u - v$

$$\frac{d}{dt} w + \nu A w + B(w,v) + B(u,w) = 0, \quad L^2((0,T), V')$$

$$w_0 = u_0 - v_0 \in H. \qquad w \in L^2((0,T), V).$$

$$\frac{d}{dt}\frac{|w(t)|^2}{2} + \nu \|w\|^2 + \langle B(w,v), w\rangle_{V'} = 0$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}|w(t)|^2 + \nu\|w\|^2 = -\langle B(w,v), w\rangle$$

$$= -\int (w\cdot\nabla) v\, w \, dx$$

$$\leq \int |\underbrace{w}_{L^4} \cdot \underbrace{\nabla v}_{L^2} \underbrace{w}_{L^4}| \, dx$$

$$\leq \|w\|_{L^4}^2 \|v\|$$

$$\leq |w| \cdot \|w\| \cdot \|v\|$$

$$\leq \frac{\|v\|^2 |w|^2}{2\nu} + \frac{\nu}{2}\|w\|^2$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}|w(t)|^2 + \underbrace{\frac{1}{2}\nu\|w\|^2}_{\geq \nu\lambda_1 |w|^2} \leq \frac{\|v\|^2 |w|^2}{2\nu} \Rightarrow$$

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2 + \nu\lambda_1 |w(t)|^2 \leq \frac{\|v\|^2 |w|^2}{2\nu}$$

$\Rightarrow \quad \frac{d}{dt}|w(t)|^2 \le \frac{\|v\|^2}{2}|w|^2, \quad \to * e^{-\int_0^t \|v(\tau)\|^2 \frac{d\tau}{2}}$

$\Rightarrow \quad |w(t)|^2 \le e^{\frac{1}{2}\int_0^t \|v\|^2 d\tau} \underbrace{|w(0)|^2}_{=0 \Rightarrow \text{uniqueness.}}$

$\underbrace{\sup_{0\le t\le T} |w(t)|^2}_{L^\infty} \le e^{\frac{1}{2}\int_0^T \|v\|^2 d\tau} |w_0|^2.$

$\Rightarrow$ Lipschitz Map; so solutions depend on data in Lipschitz way.

apr-8-2025

Strong solutions to Navier Stokes, 3D. H^1

$\frac{du}{dt} + \nu Au + B(u,u) = f$, $u(0) = u_0$, $f \in H$, $u_0 \in V \underset{comp}{\hookrightarrow} H$

L^2 stronger initial.

Back to Galerkin: $u_m \in \text{span}\{\varphi_1, \dots, \varphi_m\}$

$\frac{du_m}{dt} + \nu Au_m + P_m B(u_m, u_m) = P_m f$, $u_m(0) = P_m u_0$.

$\forall T > 0$, $\|u_m\|_{L^\infty(0,T,H)} \le K_0$ $\qquad \|u_m\|_{L^2(0,T,V)} \le \bar{K}_1$

$\left\|\frac{du_m}{dt}\right\|_{L^{4/3}(0,T,V')} \le K.$ $\to$ all holds since $V \subset H$.

$\Rightarrow u \in C([0,T], V')$ $\quad u \in C([0,T]; H_{weak})$

$\|u_m\| \le$ $\quad \frac{1}{2}\frac{d}{dt}\|u_m\|^2 + \nu |Au_m|^2 + (P_m B(u_m, u_m), Au_m) = (P_m f, Au_m)$

$|(B(u_m, u_m), Au_m)| = \left|\int (u_m \cdot \nabla) u_m \cdot Au_m \, dx\right|$

$\le \int \underbrace{|u_m|}_{L^6} \underbrace{|\nabla u_m|}_{L^3} \underbrace{|Au_m|}_{L^2} dx$

$\le \|u_m\|_{L^6} \|\nabla u_m\|_{L^3} |Au_m|.$

$\le \|u_m\| \cdot \|u_m\|^{1/2} |Au_m|^{1/2} |Au_m|$

$\le \|u_m\|^{3/2} |Au_m|^{3/2}.$

$\to$ Youngs $\Rightarrow$ $\leq \frac{3}{4}\nu|Au_m|^2 + \frac{1}{4}\nu^3\|u_m\|^6$

$$\frac{1}{2}\frac{d}{dt}\|u_m\|^2 + \frac{\nu}{4}|Au_m|^2 \leq \frac{1}{4}\nu^3\|u_m\|^6 + \underbrace{|f|\,|Au_m|}_{\text{Youngs}}$$

$$\leq \frac{1}{4}\nu^3\|u_m\|^6 + \frac{c|f|^2}{\nu} + \frac{\nu}{8}|Au_m|^2 \Rightarrow$$

$$\frac{1}{2}\frac{d}{dt}\|u_m\|^2 + \frac{\nu}{8}|Au_m|^2 \leq \frac{c_1}{\nu^3}\|u_m\|^6 + \frac{|f|}{\nu^3}c_2.$$

$\to (a^3+b^3) \leq (a+b)^3$

$$\Rightarrow \frac{d}{dt}\|u_m\|^2 + \frac{\nu}{4}|Au_m|^2 \leq \underbrace{\frac{2c_1}{\nu^3}}_{C_3}\Big((\|u_m\|^2)^3 + \underbrace{\frac{c_2}{c_1}\frac{|f|^2}{\nu^2}}_{(C_4^{1/3})^3}\Big).$$

$$\leq C_3(\|u_m\|^2 + C_4)^3.$$

$$\Rightarrow \frac{d}{dt}(\|u_m\|^2 + C_4) \leq C_3(\|u_m\|^2 + C_4)^3,$$

$$Z_m(t) := u_m(t) + C_4 \Rightarrow$$

$$\frac{d}{dt}Z_m(t) \leq C_3 Z_m^3(t), \; Z_m(0) \leq \|u_0\|^2 + C_4 =: Z_0.$$

$\{t : Z_m(t) \leq 2Z_0\}$; $t^* := \sup$, $t^* > 0$

$\{\tau : Z_m(t) \leq 2Z_0 \;\; \forall t \in (0,\tau)\}$.

$\hookrightarrow$ Take sup of this set, sup $= T^*$;

Option 1: $T^* = \infty$

Option 2: $T^* < \infty$.

In case 2: $\dot{Z}_m \le 8C_3 Z_0^3$ for $0 \le t \le T^*$.

$Z_m(t) \le Z_m(0) + 8C_3 Z_0^3 t \rightarrow \forall t,\ 0 \le t \le T^*$.

$$\underset{\underset{2Z_0}{\parallel}}{Z_m(T^*)} \le Z_0 + 8C_3 Z_0^3 T^* \Rightarrow$$

$$Z_0 \le 8C_3 Z_0^3 T^* \Rightarrow$$

$$\boxed{T^* \ge \frac{1}{8C_3 Z_0^2}}$$

$\Rightarrow T_* > \frac{1}{2}\frac{1}{8C_3 Z_0^2} =: T^{**} = \frac{1}{2}T^*$, on $[0, T^{**}]$,

$$\boxed{\sup_{0 \le t \le T^{**}} Z_m(t) \le 2Z_0} \Rightarrow \|u_m(t)\|^2 + C_4 \le 2(\|u_0\|^2 + C_4)$$

$\Rightarrow \|u_m(t)\|^2 \le 2(\|u_0\|^2 + C_4)$ for $0 \le t \le T^{**}$.

$$\Rightarrow \|u_m\|_{L^\infty(0,T^{**};V)} \le K_1. \rightarrow$$ independent of m.

$\Downarrow$

$$\frac{1}{2}\frac{d}{dt}\|u_m\|^2 + \frac{\nu}{8}|Au_m|^2 \le \frac{c_1}{\nu^3}\|u_m\|^6 + \frac{|f|^2}{\nu}c_2$$

$$\le \frac{c_1}{\nu^3}K_1^6 + \frac{|f|^2}{\nu}c_2. \rightarrow \text{cst.}$$

$$\|u_m(T_{**})\|^2 + \int_0^{T^{**}} \frac{\nu}{8}|Au_m(\tau)|^2 d\tau \le \|u_0\|^2 + T^{**}\left(\frac{c_1}{\nu^3}K_1^6 + \frac{|f|^2}{\nu}c_2\right)$$

$$\Rightarrow \int_0^{T^{**}} |Au_m|^2 dt \le \|u_0\|^2 + T_{**}\left(\frac{c_1}{\nu^3} K_1^6 + \frac{|f|^2}{\nu} c_2\right)\frac{2}{\nu}$$

$$\Rightarrow \|u_m\|_{L^2(0,T^{**}, D(A))} \le K_2$$

$$\left|\frac{du_m}{dt}\right| = \left|\nu A u_m + f + P_m B(u_m, u_m)\right|$$

$$\le \nu \underbrace{|Au_m|}_{L^2(0,T^{**})} + |f| + |P_m B(u_m, u_m)|$$

$$|P_m B(u_m, u_m)|^2 \le |B(u_m, u_m)|^2$$

$$\int |u_m|^2 |\nabla u_m|^2 dx \le \|u_m\|_{L^6}^2 \|\nabla u_m\|_{L^3}^2$$

$3 \quad 3/2$

$$\le \|u_m\|^2 \|u_m\| \cdot |Au_m|.$$

$$\le K_1^3 |Au_m| \in L^2(0,T^{**}).$$

$$\Rightarrow |P_m B(u_m, u_m)| \in L^4(0,T^{**}), \Rightarrow$$

$$\left\|\frac{du_m}{dt}\right\|_{L^2(0,T^{**}, H)} \longrightarrow \le \overline{K_0'}, \text{ so } \quad D(A) \subset V \subset H$$

$$X_1 \hookrightarrow X_0 \hookrightarrow X_{-1}$$

$$u_{m_j} \xrightarrow{} u \text{ in } L^2(0,T^{**}, V),$$
Str

$$u_{m_j} \overset{w^*}{\rightarrow} u \text{ in } L^\infty(0,T^{**}, V),$$

$$u_{m_j} \rightharpoonup u \text{ in } L^2(0,T^{**}, D(A)),$$

$$\frac{d}{dt} u_{m_j} \rightharpoonup \frac{du}{dt} \text{ in } L^2(0,T^{**}, H).$$

Apr-15-2025

Weak-Strong Uniqueness, strong solution is unique in class of weak solution.

Let $u_0 = v_0 \in V$, $f \in H$. Let u be a weak solution with $u(0) = u_0$; let v be a strong solution w/ $v_0 = v(0)$.

$\qquad\qquad\qquad\qquad T \to$ Leray Hopf (Energy Inequality)

$\to$ u is global, v short time T^*.

$\to$ Claim $u(t) = v(t)$ for all $t \in [0, T^*]$.

Proof: $\dfrac{du}{dt} + \nu A u + B(u,u) = f$ in $L^{4/3}(0,T^*,V')$.

$\dfrac{dv}{dt} + \nu A v + B(v,v) = f$ in $L^2(0,T^*,H)$. $\leftarrow$ dual

$v \in C([0,T, V)$, $v \in L^\infty(0,T_*,V)$, so $v \in L^4(0,T_*,V)$.

$u \in L^\infty(0,T^*,H)$, $u \in L^2(0,T^*,V)$, so $u \in L^2(0,T^*,H)$ Test fns.

Take action:

$\langle \frac{du}{dt}, v\rangle_{V'} + \langle \frac{dv}{dt}, u\rangle + 2\nu((u,v)) + (B(u,u),v)_{V'} + (B(v,v),u)_V =$

$\langle f, u\rangle_{V'} + \langle f, v\rangle_{V'}$.

$\to$ Can show (u,v) is abs. cont, and

$\dfrac{d}{dt}(u,v) = \langle \frac{du}{dt}, v\rangle + \langle \frac{dv}{dt}, u\rangle$; then integrate:

$\overset{-2}{(u(t),v(t))} + \overset{-2}{2\nu\int_0^t ((u,v))} + \overset{-2}{\int_0^t \langle B(u,u),v\rangle} + \overset{-2}{\int_0^t \langle B(v,v),u\rangle} = \overset{-2}{\cancel{2}\int_0^t} \langle f, u + v\rangle - 2(u_0,v_0)$

→ Leray Hopf:

$$|u(t)|^2 + 2\nu \int_0^t \|u(\tau)\|^2 d\tau \underset{\downarrow}{\leq} 2\int_0^t (f, u) d\tau + |u_0|^2$$

"=" for v.

$$|v(t)|^2 + 2\nu \int_0^t \|v(\tau)\|^2 d\tau = 2\int_0^t (f, v) d\tau + |v_0|^2.$$

Add! $w = u - v$: can show

$$|w(t)|^2 + 2\nu \int_0^t \|w(\tau)\|^2 d\tau + 2\int_0^t \langle B(w, v), w\rangle_{V'} d\tau \leq |u_0 - v_0|^2$$

$$= |w_0|^2$$
$$= 0$$

$\Downarrow$

$$|w(t)|^2 + 2\nu \int_0^t \|w(\tau)\|^2 d\tau + 2\int_0^t \langle B(w, v), w\rangle d\tau \leq 0$$

$\Downarrow$

$$|w(t)|^2 + 2\nu \int_0^t \|w(\tau)\|^2 d\tau \leq \int_0^t \int_\Omega \underbrace{|w|}_{L^6} \underbrace{|\nabla v|}_{L^2} \underbrace{|w|}_{L^3} dx \, d\tau$$

bounded.

$$\leq \int_0^t \|w\| \, \|v\| \, |w|^{1/2} \|w\|^{1/2} d\tau.$$

$$\leq K_1 \int_0^t |w|^{1/2} \|w\|^{3/2} d\tau. \quad \text{youngs.}$$

$$\leq \frac{K_1}{16 \nu^3} \int_0^t |w|^2 d\tau + \frac{\nu}{2} \int_0^t \|w\|^2 d\tau$$

$$\Rightarrow |w(t)|^2 \leq \frac{16 K_1^4}{\nu^3} \underbrace{\int_0^t |w(t)|^2 d\tau}_{\xi(t)} \to \boxed{\text{Gronwall.}}$$

$$\xi'(t) \le \frac{16K_1^4}{\nu^3}\xi(t) \quad * \quad e^{-\frac{16K_1^4}{\nu^3}t}.$$

$$\left(\xi(t)\, e^{-\frac{16K_1^4}{\nu^3}t}\right)' \le 0 \Rightarrow$$

$$\xi(t) \le e^{\frac{-16K_1^4}{\nu^3}t}\,\Psi(0) = 0.$$

$$\Rightarrow \int_0^t \|w(\tau)\|^2 d\tau \le 0 \Rightarrow$$

$$\int_0^{T^*} |w(\tau)|^2 d\tau = 0 \Rightarrow w(\tau) = 0 \text{ a.e.}$$

$\rightsquigarrow$ now go to 2D. WTS global existence of strong solution in 2D.

$$\frac{du}{dt} + \nu Au + B(u,u) = f \quad \text{in } L^2(0,T,H), \quad \underbrace{Au \in L^2(0,T,H)}_{\text{for strong.}}$$

Formally:

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \nu|Au|^2 = -(B(u,u), Au) + (f, Au)$$

$$\le |(\underset{L^4}{B(u},\underset{L^4}{u)},\underset{L^2}{Au})| + |(f, Au)|.$$

$$\le \|u\|_{L^4}\|\nabla u\|_{L^4}\|Au\|_{L^2} + |f||Au|, \quad \|u\|_{L^4} \le C|u|^{1/2}\|u\|^{1/2}$$

$$\overset{\nearrow \text{Bound}}{\le} |u|^{1/2}\|u\|^{1/2}\|u\|^{1/2}|Au|^{1/2}|Au|^{1/2} + |f||Au|. \Rightarrow$$

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \nu|Au|^2 \le K_0^{1/2}\|u\|\cdot|Au|^{3/2} + |f||Au|$$

$$\le \frac{K_0^2}{\nu^3}\|u\|^4 + \frac{\nu}{4}|Au|^2 + \frac{\nu}{4}|Au|^2 + \frac{|f|^2}{\nu} \Rightarrow$$

$$\frac{d}{dt}\|u\|^2 + \nu|Au|^2 \le \frac{32 K_0^4}{\nu^3}\|u\|^4 + \frac{|f|^2}{\nu}$$

$$\le \frac{32K_0^4}{\nu^3}\left(\|u\|^4 + \frac{|f|^2}{\nu}\frac{\nu^3}{32K_0^4}\right)$$

$$\le C_1\left(\|u\|^2 + C_2\right)^2,$$

$$\frac{d}{dt}\Big(\underbrace{\|u\|^2 + C_2}_{Z(t)}\Big) \le C_1\left(\|u\|^2 + C_2\right)^2.$$

$$\boxed{\dot{Z}(t) \le C_1 Z^2.}$$

in 2D, bounded.

$$\int_0^T Z(t)\,d\tau = \int_0^T \|u\|^2 d\tau + C_2 T$$

$$\le \overline{K_1} + C_2 T.$$

$$\dot{Z} \le C_1 Z \cdot Z$$

$$Z(t) \le Z(0)\, e_1^{c_1 \int_0^t Z(\tau)d\tau}$$

$$\boxed{Z(t) \le Z(0)\, e^{c_1(\overline{K_1} + C_2 T)}}$$

Bounded.

Apr-17-25 | Euler Equations

→ NS w/ no viscosity

→ Numerically, Euler is easier to solve.

→ But, theoretically still many questions about convergence of NS to Euler as $\nu \to 0$.

$$\varepsilon \ll 1$$

Consider $\begin{cases} \dot{x} = f(x,y) \\ \varepsilon\dot{y} = g(x,y) \end{cases}$, as $\varepsilon \to 0$, $\dot{y} \to \infty$, and so need $\Delta t \to 0$ numerically.

If we take $\varepsilon = 0$, $\begin{cases} \dot{\bar{x}} = f(\bar{x},\bar{y}) \\ g(\bar{x},\bar{y}) = 0 \end{cases}$

→ If g is invertible, $\bar{y} = \varphi(\bar{x})$, say, then

$\dot{\bar{x}} = f(\bar{x}, \varphi(\bar{x}))$, $\bar{x}(0) = x_0$ → Solve this, then ask how close this is to original problem?

→ ie, $\boxed{(\bar{x}(t), \varphi(\bar{x})) \simeq (x(t), y(t)) \text{ ?!}}$

↳ eg $y_0 \simeq \varphi(x_0)$?

→ Since x is "slow", consider $\varepsilon y' = g(\xi, y)$, $\xi = x_0$, $y(0) = y_0$.

→ Euler $\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla p = 0 \\ \operatorname{div}(u) = 0 \end{cases}$ periodic, $u(0) = u_0$.

→ Galerkin, $H = L^2$ & div-free.

→ Use Stokes basis: $\partial_n v = 0$

$\partial_t u_m + P_m(u_m\cdot\nabla)u_m + \underbrace{\nabla p_m}_{0} = 0$, $\operatorname{div}(u_m) = 0$
$u_m \in H_m$.

b.c. P_m is div-free projection

$\frac{du_m}{dt} + P_m B(u_m, u_m) = 0$ → Quadratic ODE, local existence and uniqueness.

$\frac{1}{2}\frac{d}{dt}|u_m|^2 = 0 \Rightarrow |u_m(t)| = |u_m(0)| = |P_m(u_0)| \leq |u_0|$

⇒ Global existence for $\boxed{t \in \mathbb{R}}$, since energy is constant.

$\frac{1}{2}\frac{d}{dt}\|u_m\|^2 = -(P_m B(u_m, u_n), Au_m)$

$= -\int (u_m\cdot\nabla)u_m(-\Delta u_m)\,dx$

↳ Cannot bound this term since no viscosity

→ Try $\boxed{3}$ derivatives

$\frac{d}{dt}D^3 u_m + P_m D^3((u_m\cdot\nabla)u_m) + \underbrace{P_m \nabla D^3 p_m}_{0} = 0.$ // $\times D^3 u_m$.

$\frac{1}{2}\frac{d}{dt}\|D^3 u_m\|^2_{L^2} = -\int D^3((u_m\cdot\nabla)u_m)\, D^3 u_m\,dx$

$= -\underbrace{\int ((u_m\cdot\nabla)D^3 u_m)\, D^3 u_m\,dx}_{0} + \int D^2 u_m \nabla D u_m\, D^2 u_m +$

b.c. $\int (u\cdot\nabla)v\,v\,dx = 0$ since $\operatorname{div}(u) = 0$. $\qquad \int D u_m \nabla D^2 u_m\, D^3 u_m\,dx.$

$$\frac{1}{2}\frac{d}{dt}\|D^3 u_m\|_{L^2}^2 \le C\|D^2 u_m\|_{L^3}^3 + \|Du_m\|_{L^\infty}\|Du_m\|_{L^2}^2$$

$$\le C\|D^3 u_m\|_{L^2}^3 + \|D^3 u_m\|^3, \text{ so}$$

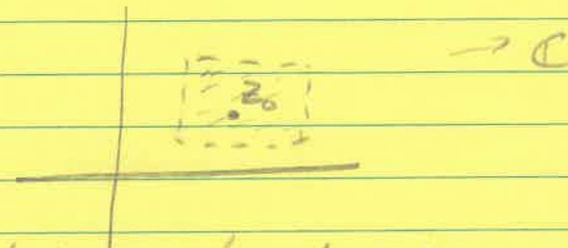
$$\dot{z}_m \le z_m^{3/2}.$$

$\hookrightarrow$ Repeat short time existence w/ H^3 boundedness.

Apr-18-2025

analyticity of NS in time:

→ Consider $\frac{dw}{dz} = F(w)$ $\quad w(z_0) = z_0$, $\quad w \in \mathbb{C}$.



→ Need to assume F is analytic about w_0, in order for problem to make sense.

→ Solving this means w is analytic.

→ Back to NS: complexify time:

ξ, $Re(\xi) > 0$, $\boxed{\text{so I want } u \text{ to be complex}}$.

$\frac{du}{d\xi}$

→ so we want $u \in H, V, u \in D(A)$, but these are Real spaces, so lets complexify:

$H_{\mathbb{C}} = H \oplus iH = \{\varphi + i\psi : \varphi, \psi \in H\}$, same for $V_{\mathbb{C}}$, $(D(A))_{\mathbb{C}}$, but now we need to understand

$\frac{du}{d\xi} + \nu Au + B(u,u) = f$ $\boxed{\text{as complex!}}$.

$\frac{du_m}{d\xi} + P_m B(u_m, u_m) + \nu A u_m = P_m f$.
$u_m(0) = P_m u_0$

$u_0 \in V \subset V_{\mathbb{C}}$.

So the "F" is $\overset{\text{quad}}{B(u_m, u_m)}$, $\overset{\text{linear}}{Au_m}$, $\overset{\text{const}}{f}$, so things are analytic,

$$u_m = \sum_1^m \alpha_k(\xi)\, \varphi_k(x).$$

So, since we have analytic F, we can apply the same idea and get analytic in time (at least in a neighborhood).

Apr-22-2025

Have to focus on 2D to get long time behavior.

Dynamical System:

$$u(t) = \sum_{1}^{\infty} \alpha_k(t) w_k; \quad \{\alpha_k(t)\}_{k=1}^{\infty}$$

Dissipative:

→ Find a ball (absorbing) such that all solutions end up in the ball.

$$\frac{du}{dt} + \nu A u + B(u,u) = f,$$

$$\frac{d}{dt}\frac{|u|^2}{2} + \nu\|u\|^2 = |fu| \le |f|\cdot|u| \le \frac{|f|\cdot\|u\|}{\lambda_1^{1/2}}$$

$$\Rightarrow \frac{d}{dt}|u|^2 + \nu\|u\|^2 \le \frac{|f|}{\nu\lambda_1}.$$

$$\frac{d}{dt}|u|^2 + \nu\lambda_1|u|^2 \le \frac{|f|^2}{\nu\lambda_1} \Rightarrow$$

$$|u(t)|^2 \le e^{-\nu\lambda_1 t}|u_0|^2 + \frac{|f|^2}{(\nu\lambda_1)^2}\left(1 - e^{-\nu\lambda_1 t}\right) \Rightarrow$$

$$\limsup_{t\to\infty} |u(t)|^2 \le \frac{|f|^2}{(\nu\lambda_1)^2} =: R_0^2$$

If $|u_0| < 2R_0$, then $|u(t)|^2 \le \underbrace{e^{-\nu\lambda_1 t}4R_0^2}_{} + \frac{|f|^2}{(\nu\lambda_1)^2}\left(1 - e^{-\nu\lambda_1 t}\right)$

$$\le e^{-\nu\lambda_1 t}(4R_0^2) + (4R_0^2)\left(1 - e^{-\lambda_1\nu t}\right)$$

$$\le 4R_0^2.$$

↳ Stays in this ball.

$s_1=0, s_1=1$

$s_1 \quad d/2 = s^* \quad s_2$

dn 2D, $2/2 = 1 = s^*$ $\qquad \|u\|_{L^\infty} \le C \|u\|_{L^2}^{1/2} \|u\|_{H^2}^{1/2}$

dn 3D, $3/2 = s^*$ $\qquad \|u\|_{L^\infty} \le \begin{cases} C\|u\|_{H^1}^{1/2} \|u\|_{H^2}^{1/2} \\ C\|u\|_{L^2}^{1/4} \|u\|_{H^2}^{3/4} \end{cases}$

This are the only exponents that give correct scaling, so best possible bounds.

• Interpolation:

2D: $q=4$, $\frac{1}{q} = \frac{1}{2} - \frac{s}{2} \Rightarrow s = 1/2, \Rightarrow \|u\|_{L^4} \le C\|u\|_{H^{1/2}} \le C\|u\|_{L^2}^{1/2}\|u\|_{H^1}^{1/2}$.

3D: $q=4 \Rightarrow \frac{1}{q} = \frac{1}{2} - \frac{s}{3} \Rightarrow s = \frac{3}{4} \Rightarrow \|u\|_{L^4} \le C\|u\|_{H^{3/4}} \le C\|u\|_{L^2}^{1/4}\|u\|_{H^1}^{3/4}$.

$q=3 \Rightarrow \|u\|_{L^3} \le C\|u\|_{H^{1/2}} \le C\|u\|_{L^2}^{1/2}\|u\|_{H^1}^{1/2}$

$q=6 \Rightarrow \|u\|_{L^6} \le C\|u\|_{H^1}$

Whole problem w/ 3D NS: Stronger norm has more power. But since the bounds are solid, this is the best we can do with this approach.

→ There are Ky inequalities, and also holds for periodic case.

Back to distributions: $\Omega = \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$,
$\mathcal{D}(\Omega) = (C_p^\infty(\Omega), \mathcal{T})$, $\mathcal{T}(\Omega)$ dense in $\mathcal{D}(\Omega)$.
↳ Trig polys

$f \in L^p(\Omega)$, $\hat{f}_k = \frac{1}{L^d}\int_\Omega f(x)\, e^{-\frac{2\pi i k\cdot x}{L}}\, dx$.

$$(f,g) = \sum_{k\in\mathbb{Z}^d} \hat{f}_k \overline{\hat{g}_k}.$$

Def 2

Let $\mu \in \mathcal{D}'(\Omega)$. $\hat{\mu}_k := \langle \mu, e^{-\frac{2\pi i k\cdot x}{L}} \rangle \frac{1}{L^d}$

Claim: Let $\varphi \in \mathcal{D}(\Omega)$, $\varphi(x) = \sum_{k\in\mathbb{Z}} \hat{\varphi}_k e^{\frac{2\pi i k\cdot x}{L}}$, $\mu \in \mathcal{D}'(\Omega)$

$$\langle \mu, \varphi \rangle = \sum_{k\in\mathbb{Z}^d} \hat{\mu}_k \overline{\hat{\varphi}_k}.$$

Proof: $S_N(\varphi) = \sum_{|k|<N} \hat{\varphi}_k e^{\frac{2\pi i k\cdot x}{L}} \to \varphi$ in $\mathcal{D}(\Omega)$!

$\langle \mu, S_N(\varphi)\rangle = \sum_{|k|<N} \hat{\mu}_k \overline{\hat{\varphi}_k}$, take lim! $\Rightarrow$

$\langle \mu, S_N(\varphi)\rangle \longrightarrow \langle \mu, \varphi\rangle \Rightarrow \langle \mu, \varphi\rangle = \sum_{k\in\mathbb{Z}^d} \hat{\mu}_k \overline{\hat{\varphi}_k}$. ∎

$H^1_p(\Omega)$, $(H^1_p)^* =$ dual of H^1_p. Let $\mu \in (H^1)^* \Rightarrow \exists\, \tilde{\mu} \in \mathcal{D}'(\Omega)$ s.t. $\langle \mu, \varphi\rangle_{(H^1)^*\times H^1} = \langle \tilde{\mu}, \varphi\rangle\ \forall \varphi \in \mathcal{D}(\Omega)$. $\to$ Claim

Proof: If $\varphi \in \mathcal{D}(\Omega) \Rightarrow \varphi \in H^1(\Omega)$. $(\mathcal{D}(\Omega) \subset H^1(\Omega))$
So $\langle \mu, \varphi\rangle_{(H^1)^*\times H^1}$, define $\langle \tilde{\mu}, \varphi\rangle := \langle \mu, \varphi\rangle_{(H^1)^*\times H^1}$.

Linearity is clear. Let $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega)$. Show $\langle \tilde{\mu}, \varphi_n\rangle \to \langle \tilde{\mu}, \varphi\rangle$. But $\varphi_n \to \varphi$ in $\mathcal{D}(\Omega) \Rightarrow$ $\varphi_n \to \varphi$ in H^1. And thats it. ∎

Thus: $(H^1_p(\Omega))^* \subset \mathcal{D}'(\Omega)$.

Jan-30-2025

Recap: $\mu \in \mathcal{D}'$, $\varphi \in \mathcal{D}$, $\langle \mu, \varphi \rangle = \left(\sum_{k \in \mathbb{Z}^d} \tilde{\mu} \overline{\tilde{\varphi}} \right) L^d$.

If $\mu \in (H^1)'$, $\exists\ \tilde{\mu} \in \mathcal{D}'$ s.t. $\langle \mu, \varphi \rangle = \langle \tilde{\mu}, \varphi \rangle \ \forall \varphi \in \mathcal{D}$,

Focus on H^1, $(H^1)'$. Let $\mu \in (H^1)' \subset \mathcal{D}'(\Omega)$,

$$\tilde{\mu}_k = \langle \tilde{\mu}, e^{\frac{-2\pi i x \cdot k}{L}} \rangle = \langle \mu, e^{\frac{-2\pi i x \cdot k}{L}} \rangle_{(H^1)' \times H^1} L^d.$$

So $\langle \mu, \varphi \rangle_{(H^1)' \times H^1} = L^d \sum \hat{\mu}_k \overline{\hat{\varphi}_k} \quad \forall \varphi \in H^1$,

say $\mu \sim \sum \mu_k e^{\frac{2\pi i k \cdot x}{L}}$,

$$\|u\|_{(H^1)'} = \sup_{\|\varphi\|_{H^1} = 1} \langle \mu, \varphi \rangle_{(H^1)' \times H^1} = \sup_{\|\varphi\| = 1} \left(\sum_k \hat{\mu}_k \overline{\hat{\varphi}_k} \right) L^d \le$$

$$\sup_{\|\varphi\| = 1} \sum_k \left(|\hat{\mu}_k| |\hat{\varphi}_k| \frac{(1+|k|^2)^{1/2}}{L^{1/2}} \frac{L^{1/2}}{(1+|k|^2)^{1/2}} \right) L^d$$

$$\le \left(\sum |\mu_k|^2 (1+|k|^2) \right)^{1/2} L^{1/2} \underbrace{\left(\sum (1+|k|^2) |\hat{\varphi}_k|^2 \right)^{1/2} L^{1/2}}_{\|\varphi\|_{H^1} = 1}$$

$$= \left(\sum \frac{|\mu_k|^2}{(1+|k|^2)} \right)^{1/2} L^{1/2}$$

→ Ex: Let $b_k \in \mathbb{C}$, $\bar{b}_k = b_{-k}$, $\exists\ \mu \in (H^1)'$ s.t. $\hat{\mu}_k = b_k$ iff $\sum_k \frac{|b_k|^2}{1+|k|^2} < \infty$.

↳ $\langle \mu, \varphi \rangle = \sum_k b_k \overline{\hat{\varphi}_k}$.

NS: $\begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla p = f - \nu\Delta u \\ \operatorname{div}(u) = 0 \\ u\text{-periodic} \end{cases}$

$$\bar{u} = \frac{1}{|\Omega|}\int_\Omega u(x)\,dx\,;$$

Component $k = 1, 2, \dots, d$

$$\partial_t u_k + \underbrace{\sum_\ell \left(u_j \frac{\partial}{\partial x_j}\right) u_k}_{\sum \partial_j (u_j u_k)\text{; so,}} + \partial_x p - \nu \sum_\ell \partial_\ell^2 u_k = f_k$$

$\frac{d}{dt}\bar{u}_k = \bar{f}_k$. If $\bar{f}_k = 0 \Rightarrow \bar{u}_k(t) = \bar{u}_k(0)\ (=0)$.

If $\bar{f} = 0$, $\bar{u}_0 = 0$, then $\bar{u} \equiv 0\ \forall t$. So we focus on case $\bar{f} = 0$, $\bar{u}_0 = 0$, so $\bar{u}(t) = 0$.

Poincaré ineq: Let $\varphi \in H^1_p$,

$$\|\varphi - \bar{\varphi}\|_{L^2} \leq C_p^{(\Omega)} \|\nabla\varphi\|_{L^2}.$$

So $\bar{\varphi} = 0 \Rightarrow \|\varphi\|_{L^2} \leq C_p \|\nabla\varphi\|_{L^2}. \Rightarrow$

Dirichlet Norm

$$(L^{-1} C_p(\Omega)^{-1} + 1)\|\nabla\varphi\| \geq \|\varphi\|_{H^1} = L_p^{-1}\|\varphi\|_{L^2} + \|\nabla\varphi\|_{L^2} \geq \|\nabla\varphi\|_{L^2}$$

Let $X \subset L^1_p(\Omega)$. Define $\dot{X} := \{\varphi \in X : \int \varphi\,dx = 0\}$.

$\downarrow$
i.e. L^2

$\mu \in (H^1)'$, $\|\mu\|^2_{(H^1)'} = L^d \sum \frac{|\hat{\mu}_k|^2}{1+|k|^2}$; $v \in (H^1)'$

$$(\mu, v)_{H^{-1}} = L \sum_k \frac{\hat{\mu}_k \bar{v}_k}{1+|k|^2} \to \text{inner product, get}$$

$(u, \nabla p)_{(H^1)'} = 0$ ↗ divergence free. ↗ gradient of someone.

$$\Rightarrow (H^1)' = (H^1)'_\sigma \oplus (H^1)'_g$$

Sim: If $v \in (L^2_p(\Omega))^d$, $\exists!\, u \in L^2_p(\Omega)$, $p \in \dot{H}^1_p$
s.t. $v = u + \nabla p$.

$$\Rightarrow L^2_p = (L_p)_\sigma \oplus (L^2_p)_g$$

$$(\dot{L}^2_p(\Omega))^d \underset{L^2}{=} H \oplus H^\perp, \quad H = \{u \in \dot{L}^2_p(\Omega)^d : \operatorname{div}(u) = 0\}.$$

$$H^\perp = \{\nabla p \in \dot{L}^2_p(\Omega)^d : p \in \dot{H}^1_p(\Omega)\}$$

$$(\dot{H}^1_p(\Omega))^d \underset{H^1}{=} V \oplus V^\perp, \quad V = \{\varphi \in (\dot{H}^1_p(\Omega))^d : \operatorname{div}(v) = 0\}$$
$$V^\perp = \{\nabla p : p \in \dot{H}^2\}$$

Denote by $\dot{T}(\Omega)$ = trig polys w/ 0 average.

$$\mathcal{V} = \{u \in \dot{T}(\Omega) : \operatorname{div}(u) = 0\}$$

Claim:
- H is closed subspace of $\dot{L}^2$
- V is closed subspace $\dot{H}_1$
- $H = \overline{\mathcal{V}}$ in $\dot{L}^2$.
- $V = \overline{\mathcal{V}}$ in $\dot{H}^1$

- inner product in H: $(\varphi,\psi) = \int \varphi\psi \, dx \quad \varphi,\psi \in H,$
  $|\varphi|^2 = (\varphi,\varphi)$, $\rightarrow$ L^2 norm.
- inner product in V: $((\varphi,\psi)) = \int \nabla\varphi : \nabla\psi \, dx,$
  $\|\varphi\|^2 = ((\varphi,\varphi))$, norm b.c. Poincare.