PDEs II

S. Gerberding

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Contents

Chapter 1

Elements of Distribution Theory

1.1 Motivation/ Heuristic

Consider a continuous function $u : U \to \mathbb{R}$, $U \subset \mathbb{R}^n$ open.

- One characterization of u is by its point-wise values, $u(x)$; ie, $u = v$ iff $u(x) = v(x)$ for all $x \in U$.
- Equivalently, characterize u in terms of *weighted averages*: $\langle u, \phi \rangle = \int_U u(x)\phi(x)\ \mathrm{d}x$ where the test functions ϕ range over a vector space of functions on U that is (i) nice enough so that the pairing is well defined, ie, works with integral for each ϕ; (ii) rich enough so that the $\langle u, \phi \rangle = \langle v, \phi \rangle$ for all ϕ implies that $u = v$.
- The map $\phi \mapsto \langle u, \phi \rangle$ is a linear functional on that vector space.
- The weighted averages $\langle u, \phi \rangle$ make sense for a much wider class of objects than continuous functions.

Definition 1.1 (Naive)**.** A *distribution* on some open subset $U \subset \mathbb{R}^n$ is a linear functional $u : C_c^\infty \to \mathbb{R}$ that is "continuous". (Continuous insofar as we need to understand what we mean by $\phi_k \to \phi$.)

1.2 The space of test functions $C_c^\infty(U)$

Definition 1.2. $C_c^\infty(U) := \{\phi : U \to \mathbb{R} :\ \phi \text{ smooth and compact support}\}$.

Definition 1.3. Let $\phi \in C(U)$, then $\mathrm{supp}(\phi) := \overline{\{x \in U :\ \phi(x) \neq 0\}}$.

Example 1.4. The function:

$$\phi(x) := \begin{cases} e^{-\frac{1}{1-|x|^2}} & |x| < 1 \\ 0 & |x| \geq 1 \end{cases}$$

is smooth and has compact support.

Definition 1.5. Convolution:

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x-y)g(y)\ \mathrm{d}y = \int_{R^n} f(y)g(x-y)\ \mathrm{d}y$$

whenever this is well-defined.

Observe: for $f, g \in C_c(\mathbb{R}^n)$ with compact support, $\mathrm{supp}(f * g) \subset \mathrm{supp}(f) + \mathrm{supp}(g)$.

Proof. Let $x \in \mathbb{R}^n$ such that $f * g(x) \neq 0$. Then, there exist $y \in \{y : f(y) \neq 0\} \subset \mathrm{supp}(f)$ and $x - y \in \{z : g(z) \neq 0\}$, thus $x = y + (x - y) \in \mathrm{supp}(f) + \mathrm{supp}(g)$. □

Lemma 1.6 (Evans, Thm. C.6)**.** *Let $f \in C^k(\mathbb{R}^n)$, $0 \leq K < \infty$. Let ϕ be smooth with supp$(\phi) \subset \overline{B(0,1)}$ and $\int_{\mathbb{R}^n} \phi\ \mathrm{d}x = 1$. Let $\phi_\epsilon(x) := \epsilon^{-n}\phi(\frac{x}{\epsilon})$ and let $f_\epsilon := f * \phi_\epsilon$. Then the functions f_ϵ are smooth and supp$(f_\epsilon) \subset$ supp$(f) + \overline{B(0,1)}$. For $|\alpha| \leq k$, $\partial^\alpha f_\epsilon \to \partial^\alpha f$ uniformly on each compact set as $\epsilon \to 0$.*

Observe: if for $u, v \in C(U)$, we have the $\int_U u\phi\ \mathrm{d}x = \int_U v\phi\ \mathrm{d}x$ for all $\phi \in C_c^\infty(U)$, then $u = v$ pointwise everywhere, so $C_c^\infty(U)$ is rich enough.

Proof. $\int_U (u - v)\phi\ \mathrm{d}x = 0$. Suppose $w := u - v \neq 0$ at z, say $w(z) > 0$. Then, since w is continuous, it is positive on some neighborhood. Then take ϕ with support in that neighborhood, and then $\int_U w\phi\ \mathrm{d}x \neq 0$, which is a contradiction. □

Lemma 1.7 (Fundamental lemma of calculus of variations). *Let $U \subset \mathbb{R}^n$ be open $u \in L^1_{loc}(U)$. If $\int_U u(x)\phi(x)\ \mathrm{d}x = 0$ for all $\phi \in C_c^\infty(U)$. Then $u = 0$ a.e. in U.*

Definition 1.8. A sequence $\{\phi_k\} \subset C_c^\infty(U)$ converges to $\phi \in C_c^\infty(U)$ if there exists $K \subset U$ compact such that $\mathrm{supp}(\phi_k)$, $\mathrm{supp}(\phi) \subset K$ and for all $\alpha \in \mathbb{N}_0$, $\lim_{k\to\infty} \sup_{x\in K} |\partial^\alpha \phi_k(x) - \partial^\alpha \phi(x)| = 0$.

Definition 1.9. A distribution on U is a linear function $u : C_c^\infty(U) \to \mathbb{R}$ that is continuous in the following sense: for any $\{\phi_k\}$ such that $\phi_k \to \phi$ as above, then $u[\phi - k] \to u[\phi]$.

Note 1.10.

- Notation: $D'(U)$: space of distributions on U.
- $u[\phi] = \langle u, \phi \rangle$.

Example 1.11.

- locally integrable functions: any function $u \in L^1_{\mathrm{loc}}(U)$ defines a distribution by $\langle u, \phi \rangle := \int_U u\phi\ \mathrm{d}x$, $\phi \in C_c^\infty(U)$. Concrete example: $u(x) := \frac{1}{|x|^r}$, $x \in \mathbb{R}$ is in L^1_{loc} for $r < n$.
- Dirac delta function δ_x: $\langle \delta_x, \phi \rangle := \phi(x)$ for some fixed $x \in U$.
- (Signed) locally finite Borel measures: Any such measure μ on U defines a distribution by $\langle u, \phi \rangle = \int \phi(x0\ \mathrm{d}\mu(x)$.

Lemma 1.12. *A linear functional $u : C_c^\infty(U) \to \mathbb{R}$ is a distribution iff it is bounded in the following sense: For any compact set $K \subset U$, there exists $N \in \mathbb{N}$ and a constant $C_{K,N}$ such that for all $\phi \in C_c^\infty(U)$ with supp$(\phi) \subset K$, the following holds:*

$$|\langle u, \phi \rangle| \le C_{N,N} \sum_{|\alpha| N} \sup_{x \in K} |\partial^\alpha \phi(x)|$$

Proof. HW □

Definition 1.13. Let $u \in D'(U)$. If there is an integer N such that u is bounded for all compact $K \subset U$ and some $C \equiv C(K)$, then we say that u has order $\le N$.

Example 1.14.

When $u \in L^1_{loc}$, $|\langle u, \phi \rangle| \le \|u\|_{L^1(K)} \|\phi\|_\infty$, so the order is 0.

Definition 1.15. We say that $u \in D'(U)$ vanishes in an open subset $V \subset U$ if $\langle u, v \rangle = 0$ for all $\phi \in C_c^\infty(U)$ with $\mathrm{supp}(\phi) \subset V$.

Definition 1.16. Let $V_{max} := \bigcup\{V :\ V \subset U\text{open and } u \text{ vanishes in } V\}$. Then the *support of* u, denoted by supp(u), is the complement of V_{max}.

1.3 Operations with Distributions

Basic principle: Adjoint method and approximation method.

1.3.1 Adjoint Method

An operation T on smooth functions is generalized to $D'(U)$ by computing the adjoint operation T' with the property:

$$\int_U (Tu)\phi\ \mathrm{d}x = \int_U u(T'\phi)\ \mathrm{d}x \forall \phi \in C_c^\infty(U),\ \forall u \in C_c^\infty(U)$$

such that $T'\phi \in C_c^\infty(U)$. (In this sense, we can plug $T'\phi$ in as a test function.)

Then defining $\langle Tu, \phi \rangle := \langle u, T'\phi \rangle$.

Example 1.17.

- Multiplication by a smooth function:$u, \phi \in C_c^\infty$. For $f \in C_c^\infty(U)$:

$$\int fu\phi\ \mathrm{d}x = \int uf\phi\ \mathrm{d}x.$$

  So we define $\langle fu, \phi \rangle := \langle u, f\phi \rangle\ \forall \phi C_c^\infty(U)$.

- Differentiation: $\int \partial_j u \phi\ \mathrm{d}x = -\int u \partial_j \phi\ \mathrm{d}x$ (IBP). Given $u \in D'(U)$, we define for $1 \le j \le n$, $\langle \partial_j u, \phi \rangle := -\langle u, \partial_j \phi \rangle$. More generally, for multi-index α, $\langle \partial^\alpha u, \phi \rangle := (-1)^{|\alpha|} \langle u, \partial^\alpha \phi \rangle$.

- Convolution with a smooth and compactly supported function: [Eskin] Given $u \in D'(\mathbb{R}^N)$ and $f \in C_c^\infty(\mathbb{R}^n)$, we define $\langle f * u, \phi\rangle := \langle u, f *' \phi\rangle$
$$\left(\int (f*u)\phi(x)\ \mathrm{d}x = \int \left(\int f(x-y)u(y)\ \mathrm{d}y \phi(x)\ \mathrm{d}x\right) = \int u(y) \underbrace{\left(f(x-y)\phi(x)\ \mathrm{d}x\right)}_{:=f*'\phi}\ \mathrm{d}y\right).$$

Lemma 1.18. *For $f \in C_c^\infty(U)$ and $u \in D'(U)$, $f * u$ is a smooth function, in fact $\partial^\alpha(f*u)(x) = ((\partial^\alpha f) * u)(x)$.*

Note 1.19. We $\langle f*u, \phi\rangle := \langle u, f *' \phi\rangle - \int_u(x) \int f(y-x)\phi(y)\ \mathrm{d}x\ \mathrm{d}x = \int \underbrace{\left(\int u(x) f(y-x)\ \mathrm{d}x\right)}_{\langle u, f(y-\cdot)\rangle} \phi(y)\ \mathrm{d}y$. The expression $g(x) := \langle u, f(x-\cdot)\rangle$, $x \in \mathbb{R}^n$ defines a continuous function $g : \mathbb{R}^n \to \mathbb{R}$. Let $x_k \to x$ as $k \to \infty$. Then, $g(x_k) = \langle u, f(x_k - \cdot\rangle = \langle u, f(x-\cdot\rangle = g(x)$ because $f(x_k - \cdot) \to f(x-\cdot)$ in C_c^∞ sense.

Proof. Identify $f*u$ with a continuous function. Let $f * u := g$. We need to show $\langle f*u, \phi\rangle = \langle g, \phi\rangle$ for all $\phi \in C_c^\infty$. Recall: $\langle f * u, \phi\rangle = \langle u, f *' \phi\rangle$, $(f *' u)(x) = \int f(y-x)\phi(y)\ \mathrm{d}y$. Let $S_N(x) := \sum_{0 \le l \le N} f(\eta_l - x)\phi(\eta_l)\Delta_l$. Then $S_N(x) \to (f *' \phi)(x)$ uniformly in x as $N \to \infty$ and same for its derivatives. Thus, $S_N \to f *' \phi$ in D. Hence, $\langle f * u, \phi\rangle := \langle u, f *' \phi\rangle = \lim_{N\to\infty}\langle u, S_N\rangle = \lim_{N\to\infty} \sum_{0\le l\le N} \underbrace{\langle u, f(\eta_l - \cdot\rangle}_{g(\eta_l)} \phi(\eta_l)\Delta_l = \int g(y)\phi(y)\ \mathrm{d}y = \langle g, \phi\rangle$. Remains to check that $g = f * u$ is smooth. Lets verify $\partial_i g(x)$ exists:
$$\lim_{h\to 0} \frac{1}{h}\left(g(x+he_i) - g(x)\right) = \lim_{h\to 0}\langle u, \frac{1}{h}(f(x+he_i - \cdot) - f(x-\cdot)\rangle = \langle u, \partial_i f(x-\cdot)\rangle.$$
□

1.4 Guest Lecture: Schrodinger Equation

$$i\partial_t u + \Delta u = 0; \quad u(0,x) = u_0.$$

- Δ= self adjoint
- $i\Delta$ = skew-adjoint
- $e^{it\Delta}$ = unitary group on L^2

Definition 1.20 (Fourier)**.** Let $f \in L^1(\mathbb{R}^d)$. Then, $\mathcal{F}f)(\xi) = (2\pi)^{-d/2}\int e^{-x\cdot\xi} f(x)\ \mathrm{d}x$.

Then, the equation becomes:
$$i\partial_x \hat{u}(t,\xi) - |\xi|^2\hat{u}(t,\xi) = 0; \quad \hat{u}(0,\xi) = \hat{u}_0(\xi)$$

Thus, $\hat{u}(t,\xi) = e^{-it|\xi|^2}\hat{u}_0(\xi)$

Definition 1.21 (Inverse Fourier Transform)**.** $\mathcal{F}^{-1}g(x) = \tilde{g}(x) = (2\pi)^{-d/2}\int g(\xi)e^{ix\cdot\xi}\ \mathrm{d}\xi$

So $u(t,x) = \mathcal{F}^{-1}(e^{-it|\xi|^2}\hat{u}_0(\xi)$.
Note:
$$(2\pi)^{-d/2}\int e^{ix\cdot\xi}\hat{f}(\xi)\ \mathrm{d}\xi = (2\pi)^{-d/2}\lim_{\epsilon\to 0}\int e^{-it|\xi|^2}e^{-\epsilon|\xi|^2}e^{ix\cdot\xi}\hat{f}(\xi)\ \mathrm{d}\xi = (2\pi)^{-d/2}\lim_{\epsilon\to 0}\lim_{\epsilon\to 0}\int f(y)\int e^{i(x-y)\cdot\xi}e^{-\epsilon|\xi|^2}e^{-it|\xi|^2}\ \mathrm{d}\xi\ \mathrm{d}y.$$
$$(2\pi)^{-d}\int e^{-\epsilon|\xi|^2}e^{-it|\xi|^2}e^{i(x-y)\xi}\ \mathrm{d}\xi = (2\pi)^{-d}\int e^{-\epsilon|\xi|^2}e^{-it|\xi - \frac{x-y}{it}|^2}e^{\frac{i|x-y|^2}{4t}}\ \mathrm{d}\xi = (2\pi)^{-d/2}\int e^{-\epsilon|\xi|^2}e^{-it|\xi|^2}\ \mathrm{d}\xi$$
Not sure how we got here, but:
$$u(t,x) = (4\pi t)^{-d/2}e^{-id\pi/4}\int e^{\frac{i|x-y|^2}{4t}}u_0(y)\ \mathrm{d}y.$$
So:
$$\|e^{it\Delta}u_0\|_\infty \le \frac{1}{|4\pi t|^{d/2}}\|u_0\|_1$$
$$\|e^{it\Delta}u_0\|_2 = \|u_0\|_2.$$

Looking at:

$$i\partial_t u + \Delta u = F:$$

$$u(t) = -i\int_0^t e^{i(t-\tau)\Delta}F(\tau)\ \mathrm{d}\tau.$$

Lemma 1.22. *Suppose $(d(1/2-1/q) < 1$, $p, \tilde{p} < \infty$ and $d(1/2-1/q) = \frac{1}{\tilde{p}} + \frac{1}{o}$. Then*

$$\|\int_{-\infty}^t e^{i(t-\tau)\Delta}F(\tau)\ \mathrm{d}tau\|_{L_t^p L_x^\epsilon(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|F\|_{L_t^{\tilde{p}} L_x^{q'}(\mathbb{R}\times\mathbb{R}^d)}.$$

Proof. See book. $\square$

Lemma 1.23. *For $u \in \mathcal{D}'(U)$ and $f \in C_c^\infty(\mathbb{R}^n)$, $f * u \in C^\infty(\mathbb{R}^n)$ and $supp(f*u) \subset supp(f) + supp(u)$.*

Proof. Reminder: $(f*u)(x) = \langle u, f(x-\cdot)\rangle$. Regarding the support, let $x \in \mathbb{R}^n$ such that $(f*u)(x) = \langle u, f(x-\cdot)\rangle \neq= 0$. Then there exists a $y \in \mathbb{R}^n$ such that $x - y \in \text{supp}(f)$ and $y \in \text{supp}(u)$. Then $x = x - y + y \in \text{supp}(f) + \text{supp}(u)$. $\square$

Example 1.24 (More examples of distributions:)**.**

- Heaviside function: Let

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}.$$

  Then $H \in L^1_{loc}(\mathbb{R})$ where $H \in \mathcal{D}'(\mathbb{R})$.

  Claim: $H' = \delta_0$.

  Proof: For $\phi \in C_c^\infty(\mathbb{R})$:

$$\langle H', \phi\rangle = -\langle H, \phi'\rangle = -\int_{\mathbb{R}} H(x)\phi'(x)\ \mathrm{d}x = -\int_0^\infty \phi'(x)\ \mathrm{d}x = \phi(0) = \langle \delta_0, \phi\rangle$$

- Principal Value Distribution: Note that $x \to 1/x$ is not locally integrable on $\mathbb{R}$. Define the following to make up for this problem:

$$\langle p.v.\ 1/x, \phi\rangle := \lim_{\epsilon\to 0}\left(\int_\epsilon^\infty \frac{1}{x}\phi\ \mathrm{d}x + \int_{-\infty}^{-\epsilon}\frac{1}{x}\ \mathrm{d}x\right).$$

  Claim: $p.v.\ 1/x \in \mathcal{D}'(\mathbb{R})$. We show that this is well-defined.

  Write, for $\epsilon > 0$, $I_\epsilon = \int_\epsilon^1 \frac{1}{x}\phi(x)\ \mathrm{d}x + \int_{-1}^\epsilon \frac{1}{x}\phi(x)\ \mathrm{d}x + \int_1^\infty \frac{1}{x}\phi(x)\ \mathrm{d}x + \int_{-\infty}^{-1}\frac{1}{x}\phi(x)\ \mathrm{d}x$. Then, the first two integrals can we rewritten as:

$$\int_\epsilon^1 \frac{1}{x}(\phi(x)-\phi(0))\ \mathrm{d}x + \int_\epsilon^1 \frac{1}{x}\phi(x)\ \mathrm{d}x + \int_{-1}^{-\epsilon}\underbrace{\frac{1}{x}(\phi(x)-\phi(0))}_{\int_0^x \phi'(y)\ \mathrm{d}y}\ \mathrm{d}x + \int_{-1}^{-\epsilon}\frac{1}{x}\phi(0)\ \mathrm{d}x.$$

  Here we let $\epsilon \to 0$: then we are integrating a nice smooth function, so it always exists, and so we can take a limit, and everything is well-defined. Continuity still needs to be shown as a distribution; useful to work with boundedness characterization instead of the limit version.

1.5 Sequences of Distributions

Definition 1.25. A sequence $\{u_k\}_k \subset \mathcal{D}'(U)$ converges to $u \in \mathcal{D}'(U)$ if for every $\phi \in C_c^\infty(U)$, $\langle u_k, \phi\rangle \to \langle u, \phi\rangle$. We say that $u_k \to u$ as $k \to \infty$.

Lemma 1.26. *Let $u_k \to u$ in distribution. Then $\partial_\alpha u_k \to \partial^\alpha u$ in $\mathcal{D}'(U)$.*

Proof. Follows from definitions: $\langle \partial^\alpha u_k, \phi\rangle = (-1)^{|\alpha|}\langle u_k, \partial^\alpha\phi\rangle \to (-1)^{|\alpha|}\langle u, \partial^\alpha \phi\rangle = \langle \partial^\alpha u, \phi\rangle$. $\square$

More examples:

Example 1.27.

- Suppose $\{u_k\}_k \subset L^1_{loc}(\mathbb{R}^n)$ with $u_k(x) \to u(x)$ a.e. as $k \to \infty$. If there exists a $v \in L^1_{loc}(\mathbb{R}^n)$ such that$|u_k(x)| \leq |v(x)|$ for a.e. $x \in \mathbb{R}^n$, then $u_k \to u$ in $\mathcal{D}'(U)$.

  Proof: This is just Lebesgue's Dominated Convergence Theorem.

- Take

$$u_k(x) = \begin{cases} e^{-ikx} & x > 0 \\ 0 & x \leq 0. \end{cases}$$

Then $u_k(x)$ does not have a pointwise limit for $x \in (0, \infty)/2\pi z$, we $u_k \to 0$ in $\mathcal{D}'(U)$.

Proof: $\langle u_k, \phi \rangle = \int_0^\infty e^{ikx} \phi(x) \, \mathrm{d}x = \frac{1}{ik}\phi(0) - \int_0^\infty \frac{1}{ik} e^{ikx} \phi'(x) \, \mathrm{d}x \to 0$.

Lemma 1.28. *Let $\eta \in L^1(\mathbb{R}^n)$. Let $\eta_\epsilon(x) = \epsilon^{-n}\eta(x/\epsilon)$. Then $\eta_\epsilon \to (\int \eta)\delta_0$ in $\mathcal{D}'(\mathbb{R}^n)$.*

Proof. Let $\phi \in C_c^\infty(\mathbb{R}^n)$. Then $\langle \eta_\epsilon, \phi \rangle = \int_{\mathbb{R}^n} \epsilon^{-n} \eta(x/\epsilon)\phi(x) \, \mathrm{d}x = \int_{\mathbb{R}^n} \eta(z)\phi(\epsilon z) \, \mathrm{d}z \to_{\epsilon \to 0} \int_{\mathbb{R}^n} \eta(z)\phi(0) \, \mathrm{d}z = (\int \eta)\delta_0$. □

Proposition 1.29. Let $\eta \in C_c^\infty(\mathbb{R}^n)$ with $\int \eta \, \mathrm{d}x = 1$. For every $\epsilon > 0$ set $\eta_\epsilon(x)$ be as usual. Let $u \in \mathcal{D}'(\mathbb{R}^n)$. Then, $\eta_\epsilon * u \to u$ in $\mathcal{D}'(\mathbb{R}^n)$ as $\epsilon \to 0$.

[Approximation of distributions by smooth functions]

Proof. For $\phi \in C_c^\infty(\mathbb{R}^n)$, $\langle \eta_\epsilon * u, \phi \rangle = \langle u, \eta_\epsilon *' \phi \rangle$. Here, $\eta_\epsilon *' \phi \to \phi$ in C_c^∞. And so $\langle u, \eta_\epsilon *' \phi \rangle \to \langle u, \phi \rangle$. □

Remark 1.30. We can even approximate any $u \in \mathcal{D}'(U)$ by a sequence of C_c^∞ functions by taking $\eta_{1/k} * (\chi_k u)$, where:

$$\chi_k(x) = \begin{cases} 1 & |x| \leq k \\ 0 & |x| \geq 2k. \end{cases}$$

and smooth out in between.

1.6 Operations between distributions

- Convolution with a compact supported distribution:

  Definition 1.31.

Definition 1.32. The convolution of $u, v \in \mathcal{D}'(\mathbb{R}^n)$ with $\mathrm{supp}(v)$ compact is defined by:

$$\langle v * u, \phi \rangle := \langle u, v *' \phi \rangle \text{ with } (v *' \phi)(y) := \langle v, \phi(y + \cdot) \in C_c^\infty(\mathbb{R}^n) \rangle.$$

To show that $v * u$ is again a distribution: linearity is clear. Let $\phi_k \to \phi$ in C_c^∞, then:

$$\langle v * u, \phi_k \rangle = \langle u, v *' \phi_k \rangle \to \langle u, v *' \phi \rangle = \langle v * u, \phi \rangle.$$

To check that this convergence actually occurs, use boundedness of v.

Definition 1.33. Let $u \in \mathcal{D}'(U)$. Then the *singular support* of u is defined to be the complement of the union of all open sets on which u coincides with a smooth function, ie, singular support of u is $U \setminus \bigcup \{V : \ V \subset U \text{ open in which } u \in C^\infty(V)\}$.

Proposition 1.34. Let $u, v \in \mathcal{D}'(U)$ such that $\mathrm{singsupp}(u) \cap \mathrm{singsupp}(v) = \emptyset$. Then the product uv is well defined.

Proof. Cover U by open set $Y \subset U$ such that $u \in C_c^\infty(Y)$ or $v \in C_c^\infty(Y)$. Then define "uv":
If $u \in C^\infty(Y)$:, $\phi \in C_c^\infty(Y)$: $\langle uv, \phi \rangle := \langle v, u\phi \rangle$. If $v \in C^\infty(Y)$: for $\phi \in Cc(Y)$, $\langle uv, \phi \rangle := \langle u, v\phi \rangle$. □

Recall the following definition:

Definition 1.35. The Schwartz space $S(\mathbb{R}^n)$ consists of all complex-valued functions $\phi : \mathbb{R}^n \to \mathbb{R}$ such that, for all $\alpha, \beta \in \mathbb{N}_0^n$.

$$\sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f(x)| < \infty.$$

Example 1.36. $\phi \in Cc \implies \phi \in S$; $e^{-|x|^2} \in S$.

Definition 1.37. A sequence of Schwartz functions ϕ_k converges to $\phi \in S(\mathbb{R}^n)$ if $\sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta (\phi_k - \phi)| \to 0$ as $k \to \infty$.

Remark 1.38. $C_c^\infty \hookrightarrow S$, that is, if $\phi_k \to \phi$ in C_c^∞, then $\phi_k \to \phi$ in $S(\mathbb{R}^n)$.

Definition 1.39 (Tempered Distributions)**.** A continuous linear functional on the Schwartz space is called a *tempered distribution*. The set of all these functionals is denoted S'. By continuity, we mean:

$$\langle u, \phi_k \rangle \to \langle u, \phi \rangle \text{ if } \phi_k \to \phi \in S. \tag{1.1}$$

We say that $\{u_k\}$ of tempered distributions converges to $u \in S'$ if $\langle u_k, \phi \rangle \to \langle u, \phi \rangle$ for all $\phi \in S$.

Remark 1.40. Since $\phi_k \to \phi \in C_c^\infty \implies \phi_k \to \phi \in S$, any $u \in S' \implies u \in D'$. Since $u_k \to u \in S'$ implies $u_k \to u \in D'$, we have $S' \subset D'$. One can show: A linear functional on $S(\mathbb{R}^n)$ is a tempered distribution if and only if there exists a $C > 0$ and integers $k, m \geq 1$ such that

$$|\langle u, f\rangle| \leq C \sum_{|\alpha|\leq k,\ |\beta|\leq m} |x^\alpha \partial^\beta f(x)|.$$

If $f \in C^\infty(\mathbb{R}^n)$ and is such that $|\partial^\beta f(x)| \leq C_\beta(1+|x|)^{N_\beta}$ for all $\beta \in \mathbb{N}_0^n$ then $f\phi \in S(\mathbb{R}^n)$ for any $\phi \in S^(\mathbb{R}^n)$.
To define tempered distributions from L^1_{loc} functions, we need some control on growth; $\int |u(x)||(1+|x|^{-N}\ \mathrm{d}x < \infty$.

1.7 Fourier transforms of tempered distributions

Recall: The Fourier Transform of an L^1 function $f \in L^1(\mathbb{R}^n) \to \mathbb{C}$ is:

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\ \mathrm{d}x.$$

Key properties:

- $\mathcal{F}[\partial_j f](\xi) = i\xi_j \mathcal{F}[f](\xi)$
- $\mathcal{F}[x_j f](\xi) = i\partial_{\xi_j}\mathcal{F}[f](\xi)$
- $\mathcal{F} : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$
- Fourier inversion theorem: $f(x) = \frac{1}{(2\pi)^{n/2}} \int e^{ix\cdot\xi}\hat{f}(\xi)\ \mathrm{d}\xi =: \mathcal{F}^{-1}[\hat{f}](x)$.
- Show: If $\phi_k \to \phi \in S(\mathbb{R}^n)$ then $\sup_\xi |\xi^\alpha \partial_\xi^\beta(\hat{\phi}_k(\xi) - \hat{\phi}(\xi)| \to 0$ as $k \to \infty$, when $\hat{\phi}_k \to \hat{\phi}_k \in S$. When, $\mathcal{F} : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ is a continuous one to one map.

Definition 1.41. The Fourier transform $\hat{u}$ of a tempered distribution $u \in S'$ is defined by:

$$\langle \hat{u}, \phi\rangle := \langle u, \hat{\phi}\rangle\ \forall \phi \in S(\mathbb{R}^n).$$

Example 1.42 (Computing Fourier transforms of tempered distributions)**.**

- $\hat{\delta_0} = (2\pi)^{-n/2}$.
- $\hat{1} = (2\pi)^{n/2}\delta_0$.
- $\hat{e^{ix\cdot()}} = (2\pi)^{n/2}\delta_x$.
- $\hat{\partial_j \delta_0} = (2\pi)^{-n/2} i x_j$

Lemma 1.43. *For $u \in S'$, $\check{\hat{u}} = u$, ie, $\mathcal{F}$ maps S' onto S'.*

Proof. $\langle \check{\hat{u}}, \phi\rangle = \langle u, \check{\hat{\phi}}\rangle = \langle u, \phi\rangle$. □

Lemma 1.44. *For any tempered distribution $u \in S'$ and any multi-index α,*

$$\hat{\partial^\alpha u} = (i\xi)^\alpha \hat{u}.$$

Proof. Generalization of the previous computations. □

Proposition 1.45. Let $u \in S'$ and $f \in S$, then $f * u$ is a C^∞ function. Moreover, for all $\alpha \in \mathbb{N}_0^n$

$$|\partial^\alpha (f * u)(x)| \leq C_\alpha (1+|x|)^{k_\alpha}$$

for some $C_\alpha > 0$ and $k_\alpha \in \mathbb{N}$.

Example 1.46 (Extending identities of FT from S to S')**.** For $f \in S$, $u \in S'$,

$$f \hat{*} u = (2\pi)^{n/2}\hat{f} \cdot \hat{u}$$

Proof. Let $\phi \in S$. Then:

$$\langle f \hat{*} u, \phi\rangle = \langle f * u, \hat{\phi}\rangle = \langle u, f *' \hat{\phi}\rangle.$$

By direct computation, $f *' \hat{\phi} = (2\pi)^{n/2}\mathcal{F}[\hat{f}\cdot\phi]$. So:

$$\langle u, f *' \hat{\phi}\rangle = \langle u, (2\pi)^{n/2}\mathcal{F}[\hat{f}\cdot\phi]\rangle = \langle \hat{u}, (2\pi)^{n/2}\hat{f}\cdot\phi\rangle = \langle (2\pi)^{n/2}\hat{f}\hat{u}, \phi\rangle.$$

□

1.8 Fundamental Solutions

Definition 1.47. Let $P = \sum_{|\alpha|\leq k} a_\alpha \partial^\alpha \ \alpha \in \mathbb{N}_0^n,\ k \in \mathbb{N}$ be a constant coefficient differential operator on $\mathbb{R}^n$. A *distribution* $E_0 \in \mathcal{D}'$ is called a fundamental solution for P is $PE_0 = \delta_0$ in $\mathcal{D}'$.

Remark 1.48. A fundamental solution E_0 is not necessarily unique since for any $u \in \mathcal{D}'$ such that $Pu = 0$, we also have that $P(E_0 + u) = \delta_0$.

Proposition 1.49. Let E_0 be a fundamental solution for a constant coefficient PDO P. Let $f \in C_c^\infty(\mathbb{R}^n)$. Then, $E_0 * f \in C^\infty$ is a smooth solution to $P(E_0 * f) = f$.

Proof. Thanks to earlier results, we have that $P(E_0 * f) = (PE_0) * f = \delta_0 * f = f$. □

Example 1.50.

- A fundamental solution for ∂_x on $\mathbb{R}$: $\partial_x E_0 = \delta_0$: $E_0 = H$, Heaviside.
- Fundamental solution for $-\Delta$ on $\mathbb{R}^n$, $n \geq 3$. $E_0 = \Phi_n = \frac{1}{(n-2)|S^{n-1}|}|x|^{-(n-2)}$; $-\Delta\Phi_n = \delta_0$.

Remark 1.51 (Alternative derivation of formula for $\Phi_n(x)$ $n \geq 3$ using Fourier Transform on S')**.** We look for Φ_n such that $-\Delta\Phi_n = \delta_0$. Take Fourier Transform for tempered distributions. $-\hat{\Delta\Phi_n} = \hat{\delta_0} = 2\pi^{-n/2}1$ and $-\Delta\hat{\Phi}_n(\xi) = |\xi|^2\hat{\Phi_n}$. Thus $\hat{\Phi_n}(\xi) = (2\pi)^{-n/2}|\xi|^{-2}$. Thus, $\Phi_n = (2\pi)^{-n/2}\mathcal{F}^{-1}[|\xi|^{-2}]$ on $\mathbb{R}^n - \{0\}$. It can be shown that $\mathcal{F}[|\xi|^{-2}](x) = 2^{n/2-2}|x|^{-(n-2)}$.

Definition 1.52. $\Gamma(s) := \int_0^\infty t^{s-1}e^{-t}\ \mathrm{d}t$.

Proposition 1.53. Let $h_a(x) = |x|^{-a}$, $x \in \mathbb{R}^n - \{0\}$ for $0 < a < n$. Then, $\hat{h_a}(\xi) = 2^{n/2-a}\Gamma(\frac{n-a}{2})/\Gamma(a/2)|\xi|^{-(n-a)}$ in the sense of $L^1 + L^2$ Fourier transforms if $n/2 < a < n$ and in the sense of S' for $0 < a < n$.

Proof. □

1.9 Sobolev Spaces

<u>Motivation:</u> Useful function space settings to use functional analysis tools for studying PDEs.

Definition 1.54. Let $U \subset \mathbb{R}^n$ open. Let $k \in \mathbb{N}_0$ and $1 \leq p \leq \infty$. We define:

$$W^{k,p}(U) := \{u \in D'(U) :\ \partial^\alpha u \in L^p(U) \text{ for every } \alpha \in N_0^n \text{ with } |\alpha| \leq k\}.$$

We equip with the norm:

$$\|u\|_{W^{k,p}(U)} := \begin{cases} \left(\sum_{|\alpha|\leq k} \|\partial^\alpha u\|_{L^p(U)}\right)^{\frac{1}{p}} & 1 < p < \infty \\ \sum_{|\alpha|\leq k} \|\partial^\alpha u\|_{L^\infty(U)}. \end{cases}$$

- Clearly, any $\phi \in C_c^\infty(U)$ belongs to every $W^{k,p}(U)$.
- As usual, we say that $u_m \to u$ in $W^{k,p}(U)$ if $\|u_m - u\|_{W^{k,p}(U)} \to 0$.

Definition 1.55. Let $U \subset \mathbb{R}^n$ be open. We denote by $W^{k,p}(U)$ the closure of $C_c^\infty(U)$ in $W^{k,p}(U)$, ie:

$$W_0^{k,p}(U) := \{u \in W^{s,p}(U) : \exists\{\phi_m\}_m \subset C_c^\infty(U)\ s.t.\ \|u - \phi_m\|_{W^{k,p} \to 0}\}.$$

Remark 1.56. We $p = 2$, $\|\cdot\|_{W^{k,2}}$ is derived from an inner produce $\|u\|_{W^{k,2}} = \langle u, u\rangle_{W^{k,p}}$, where $\langle u, v\rangle := \sum_{|\alpha|k}\int_U(\partial^\alpha u)(\partial^\alpha v)\ \mathrm{d}x$. We denote $H^k(U) := W^{k,2}(U)$, $H_0^k(U) := W_0^{k,2}(U)$.

Remark 1.57 (On the definition of a *weak derivative* as in Evans)**.** . Recall that for a distribution $u \in D'(U)$, $\langle\partial^\alpha, \phi\rangle := (-1)^{|\alpha|}\langle u, \partial^\alpha\phi\rangle$.

Definition 1.58. Suppose $u \in L^1_{loc}(U)$, $\alpha \in N_0^n$. We say that $v \in L^1_{lov}(U)$ is the α-th weak derivative of u, denoted $v = \partial^\alpha u$, provided:

$$\int_U v\phi\ \mathrm{d}x = (-1)^{|\alpha|}\int_U u\partial^\alpha\phi\ \mathrm{d}x \quad \forall\phi \in C_c^\infty(U).$$

Example 1.59. Let $U = (-1, 1)$ and $u(x) := |x|$.

1. For $\phi \in C_c^\infty(U)$, we compute:

$$\int_{-1}^{1} u(x)\phi'(x)\ \mathrm{d}x = \int_0 1^x \phi'(x)\ \mathrm{d}x + \int_{-1}^{0} (-x)\phi'(x)\ \mathrm{d}x = \cdots = -\int_{-1}^{1} \operatorname{sign}(x)\phi(x)\ \mathrm{d}x$$

Thus $\frac{\mathrm{d}}{\mathrm{d}x}(|x|) = \operatorname{sign}(x)$ in $D'(U)$ and as a weak derivative.

2.
$$\int_{-1}^{1} u(x)\phi''(x)\ \mathrm{d}x = \int_{-1}^{1} \operatorname{sign}(x)\phi'(x)\ \mathrm{d}x = \cdots = 2\phi(0) = 2\langle \delta_0, \phi\rangle.$$

Thus! $u(x) = |x|$ is not twice weakly differentiable, even thought it has all distributional derivatives *by definition*.

Remark 1.60. For existence of weak derivatives: "corners are fine, jumps are not."

Remark 1.61. • weak derivatives, if they exist, are unique

- for a $C^k(U)$ function, its ordinary derivative coincide with their corresponding weak derivatives.
- Follows from fundamental lemma of calculus of variations

Proposition 1.62 (Elementary properties of weak derivatives)**.** Assume $u, v \in W^{k,p}$, $|\alpha| \le k$. Then:

- $\partial^\alpha u \in W^{k-|\alpha|,p}(U)$ and $\partial^\beta(\partial^\alpha u) = \partial^\alpha(\partial^\beta u) = \partial^{\alpha+\beta}$
- For each $\lambda, \mu \in \mathbb{R}$, $\lambda u + \mu v \in W^{k,p}(U)$ and $\partial^\alpha(\lambda u + \mu v) = \lambda \partial^\alpha u + \mu \partial^\alpha v$
- If V is open subset of U, then $u \in W^{k,p}(V)$
- If $\chi \in C_c^\infty(U)$, then $\chi u \in W^{k,p}(U)$ and $\partial^\alpha(\chi u) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \partial^\beta \chi \partial^{\alpha-\beta} u$.

Theorem 1.63 (Properties of Sobolev spaces)**.** *Let $U \subset \mathbb{R}^n$ open, $k \in \mathbb{N}_0$, $1 \le p \le \infty$. Then:*

1. *$W^{k,p}(U)$ is complete*
2. *A function u belongs to $H^k(\mathbb{R}^n)$ if and only if $(1+|\xi|^2)^{1/2}\hat{u}(\xi) \in L^2(\mathbb{R}^n)$ Moreover, there is a $C \ge 1$ such that $\frac{1}{C}\|u\|_{H^k}(\mathbb{R}^n) \le \|(1+|\xi|^2)^{k/2}\hat{u}(\xi)\|_{L^2(\mathbb{R}^n)} \le C\|u\|_{H^k(\mathbb{R}^n)}$.*

Sketch of proof. (2): based on the fact that $\partial_j \to i\xi_j$ under the Fourier transform.

(1): Let $\{u_m\} \subset W^{k,p}(U)$ be a Cauchy sequence. Then, $\{u_m\} \subset L^p(U)$ is Cauchy and so is $\{\partial^\alpha u_m\} \subset L^p(U)$ are also Cauchy. Since L^p is complete, $u_m \to u$ in $L^p(U)$ and $\partial^\alpha u_m \to v_\alpha \in L^p(U)$. We need to show that $\partial^\alpha u$ exists in $L^p(U)$ and $\partial^\alpha u = v_\alpha$. Let $\phi \in C_c^\infty(U)$. Then:

$$(-1)^\alpha \int u \partial^\alpha \phi\ \mathrm{d}x =\leftarrow_{L^p} (-1)^{|\alpha|} \int u_m \partial^\alpha \phi\ \mathrm{d}x = \int \partial^\alpha u_m \phi\ \mathrm{d}x \to_{L^p} \int v_\alpha \phi\ \mathrm{d}x.$$

□

1.10 Approximation by smooth functions

Goal: Approximate $u \in W^{k,p}(U)$ by a sequence of smooth function. Want to discuss three type of approximations:

- "Local approximation" by smooth functions in $W^{k,p}_{loc}(U)$
- "Global approximation" by smooth functions in $C^\infty(U) \cap W^{K,p}(U)$
- "Global approximation" up to the boundary by smooth functions $C^\infty(\overline{U}) \cap W^{k,p}(U)$.

Recall: our standard mollifier $\{\eta_\epsilon\}_{\epsilon>0}$ with $\eta_\epsilon(x) := \epsilon^{-n}\eta(x/\epsilon)$ for $\eta \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp}(\eta) \subset B(0,1)$ and $\int_{\mathbb{R}^n} \eta\ \mathrm{d}x = 1$.
Recall: $U_\epsilon := \{x \in U : \ dist(x, \partial U) > \epsilon\}$.

Theorem 1.64 (Local Approximation; Evans thm. 5.3.1)**.** *Assume $u \in W^{k,p}(U)$ for some $1 \le p < \infty$ and set $u^\epsilon := \eta_\epsilon * u$ in U. Then:*

- *$u^\epsilon \in C^\infty(U_\epsilon)$ for all $\epsilon > 0$*
- *$u^\epsilon \to u$ in $W^{k,p}_{loc}(U)$ as $\epsilon \to 0$ that is for any $V \subset\subset U$ open we have $\|u^\epsilon - u\|_{W^{k,p}(V)} \to 0$.*

Proof. • Follows from standard properties of mollifiers.

- Observe $\partial^\alpha u^e \epsilon = \eta_\epsilon * \partial^\alpha u$. Thus by L^p_{loc} approximation properties of mollifiers, $\partial^\alpha u^\epsilon \to \partial^\alpha u$ in $L^p_{loc}(U)$.

$\square$

Definition 1.65. Let $U \subset \mathbb{R}^n$. A collection $\{V_j\}_j$ of open subsets $V_j \subset U$ is called an open covering of U if $\cup_j V_j = U$. A collection $\{\chi_j\}_j$ is called a smooth partition of unity subordinate to $\{V_j\}_j$ if the following hold:

1. Each χ_j is smooth
2. $\text{supp}(\chi_j) \subset V_j$
3. For each $x \in U$, $0 \leq \chi_j \leq 1$
4. For each $x \in U$, $\sum_j \chi_j(x) = 1$, where at most finitely summands are nonzero.

Lemma 1.66 (Ruding, "Functional Analysis", thm. 6.20)**.** *Let U be a nonempty subset of $\mathbb{R}^n$ and $\{V_j\}_j$ an open covering of U. Then there exists a smooth partition of unity of U subordinate to $\{V_j\}_j$.*

Theorem 1.67 (Global approximation)**.** *Let $U \subset \mathbb{R}^n$ be open and bounded and let $u \in W^{k,p}(U)$ for some $1 \leq p < \infty$. Then, there exists a sequence of smooth functions $\{u_m\}_m \subset C^\infty(U)$ such that $\|u_m - u\|_{W^{k,p}} \to 0$ as $m \to \infty$.*

Remark 1.68. • No assumptions on the boundary are needed, ie, no smoothness of the boundary

- We also do not assert that $u_m \in C^\infty(\overline{U})$.
- Not possible for $p = \infty$: consider $U = (-1, 1)$ and $f(x) := |x|$.

Proof. Let $u \in W^{k,p}(U)$ and let $\delta > 0$. We want to find a smooth function $u_\delta \in C^\infty(U)$ such that $\|u_\delta - u\|_{W^{k,p}(U)} \leq \delta$.

- Consider an open cover $\{V_j\}_j$ of U defined by $V_j := U_{j+3} \setminus \overline{U_{j+1}}$ where $U_j := \{x \in U : \ dist(x, \partial U) > 1/j\}$.
- Choose also any open set $V_0 \subset\subset U$ such that $U = \cup_{j=0}^\infty V_j$.
- Let $\{\chi_j\}_j$ be a smooth partition of unity subordinate to $\{V_j\}_j$.
- Write $u = 1 \times u = \sum_1^\infty \chi_j u$.
- Note we can view each $\chi_j u \in W^{k,p}(U)$.
- Let $\eta \in C_c^\infty(\mathbb{R}^n)$ be a standard mollifier. For each j, we choose $\epsilon_j > 0$ small enough so that $\|\eta_{\epsilon_j} * (\chi_j u) - \chi_j u\|_{W^{k,p}(U)} \leq \frac{\delta}{2^{j+1}}$, where we use our previous local approximation result and $\text{supp}(\eta_{\epsilon_j} * (\chi_j u) \subset W_j := U_{j+4} \setminus \overline{U_j}$.
- Then, $u_\delta := \sum_{j=0}^\infty \eta_{\epsilon_j} * (\chi_j u)$ belongs to $C^\infty(U)$ since each $B(x, r) \subset \overline{B(x, r)} \subset U$ for $x \in U$, there are at most finitely many nonzero terms.
- Thus $\|u_\delta - w\|_{W^{k,p}(U)} = \|\sum_0^\infty \eta_{\epsilon_j} * (\chi_j u) - \sum_0 \chi_j u\|_{W^{k,p}} \leq \sum_1^\infty \|\eta_{\epsilon_j}(\chi_j u) - \chi_j u\|_{W^{k,p}} \leq \delta$.

$\square$

Theorem 1.69 (Global Approximation by functions smooth up to the boundary)**.** *Assume $U \subset \mathbb{R}^n$ open, bounded, and with C^1 boundary. Suppose $u \in W^{k,p}(U)$ for some $1 \leq p < \infty$. Then there exists a sequence of smooth functions $\{u_m\}_m \subset C^\infty(\overline{U})$ such that $\|u - u_m\|_{W^{k,p}} \to 0$ as $m \to \infty$.*

Proof. $\square$

1.10.1 Extensions

Goal: For $U \subset \mathbb{R}^n$ open and bounded, we want to develop a way to extend any function $u \in W^{k,p}(U)$ to a function in $W^{k,p}(\mathbb{R}^n)$.

Theorem 1.70 (Evans, sec. 5.4)**.** *Assume $U \subset (\mathbb{R}^n)$ open, bounded, with C^1-boundary. Suppose $1 \leq p < \infty$. Select a bounded open set $V \subset \mathbb{R}^n$ such that $U \subset\subset V$ ($\overline{U} \subset V$). Then there exists a bounded linear operator $E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$ such that for each $u \in W^{1,p}(U)$*

- *$Eu = u$ a.e. in U*
- *Eu has support in V*
- *$\exists C = C(p, U, V)$ such that $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}$.*

Strategy: First construct Eu for $u \in C^1)(\overline{U}($. First introduce a scheme for the half ball at a flat boundary. Then extend to bounded U with C^1 boundary via smooth partition of unity and locally flattening the boundary.

Proof. Proof of theorem Step 1: Fix $x_0 \in \partial U$. Suppose first that ∂U is flat near x^0 lying in the hyperplane where $\{x_n = 0\}$ (n-th coordinate). We may assume there exists some open ball B with center X^0 and radius $r > 0$ such that $B^+ := B \cap \{x_n geq 0\} \subset \overline{U}$ and $B^- := B \cap \{x_n < 0\}$.

Step 2: For the moment, take $u \in C^1(\overline{U})$. Define

$$\overline{u} := \begin{cases} u(x) & x \in B^+ \\ -3u(x_1, \ldots, x_{n-1}, -x_n) + 4u(x_1, \ldots, x_{n-1}, -\frac{x_n}{2}) & x \in B^- \end{cases}.$$

Step 3: we claim $\overline{u}$. Write $u^- := \overline{u}|_{B^-}$ and $u^+ := \overline{u}|_{B+}$. Show first that $\partial_{x_n} u^- = \partial_{x_n} u^+$ on $\{x_n = 0\}$. From the definition:

$$\partial_{x_n} u^-(x) = (-3)(-1)\partial_{x_n} u(x_1, \ldots, x_{n-1}, -x_n) + 4(-\frac{1}{2})(\partial_{x_n})u(x_1, \ldots, x_{n-1}, -\frac{1}{2}x_n).$$

Since $(-3)(-1) + 4(-1/2) = 1$, $\partial_{x_n} u^-|_{\{x_n=0\}} = \partial_{x_n} u^+|_{\{x_n=0}$. Since $u^+ = u^-$ on $\{x_n = 0\}$, we get

$$(\partial_{x_j} u^-|_{x_n=0} = \partial_{x_j}|_{\{x_n=0\}}.$$

Step 4: We also get: $\|\overline{u}\|_{W^{1,p}(B)} \leq C\|u\|_{W^{1,k}(B^+)}$.

Step 5: Now consider the situation where ∂U is not necessarily flat. Then, we can "flatten" ∂U locally. Let $x^0 \in \partial U$ and a small ball $\tilde{B}$ so that there exists an invertible C^1 mapping Φ with inverse Ψ such that Φ "straightens out" ∂U near x^0. Write $y = \Phi(x)$ and $x = \Psi(y)$ and for $u \in B^+$, $u'(y) = u(\Psi(y)$. Now we use steps 1-3 to extend u' from B^+ a new function $\overline{u}'$ on all of B such that $\overline{u}'$ is C^1 and $\|\overline{u}'\|_{W^{1,p}(B)} \leq C\|u'\|_{W^{1,p}(B^+)}$. Denote $W := \Psi(B)$. Converting back to the x-variables, we get an extension $\overline{u}$ of u to W and we can check that $\|\overline{u}\|_{W^{1,p}(W)} \leq \tilde{C}\|u\|_{W^{1,p}(U)}$. Here, note that $\det(D\Phi) = 1$ and $\det(D\Psi) = 1$.

Step 6: Since ∂U is compact, there exists finitely many points $x^{0,i}$, $1 \leq i \leq N$ and open sets W^i and extensions $\overline{u}_i$ of u to W_i such that $\partial U \subset \cup_1^N W_i$. Take $W_0 \subset\subset U$ so that $U \subset \cup_0^N W_i$ and let $\{\chi_i\}_0^N$ be a smooth partition of unity subordinate to the open cover $\{W_i\}$. We set $\overline{u} := \sum_1^N \chi_i \overline{u}_i$ where $\overline{u}_0 := u$. Then by step 5, $\|\overline{u}\|_{W^{1,p}} \leq C\|u\|_{W^{1,p}(U)}$. Furthermore, we can arrange for the support of $\overline{u}$ to lie within V.

Step 7: For $u \in C^1(\overline{U})$ we write $Eu := \overline{u}$. Note that E is linear. Now, for arbitrary $u \in W^{1,p}(U)$, we can pick a sequence $\{u_m\} \subset C^\infty(\overline{U}) \subset C^1(\overline{U})$ such that $\|u_m - u\|_{W^{1,p}(U)} \to 0$. Then by linearity, $\|Eu_m - Eu_l\|_{W^{1,p}(\mathbb{R}^n} \leq C\|u_m - u_l\|_{W^{1,p}(U)}$. Thus we have a Cauchy sequence, so$\{Eu_m\}_1^\infty$, and so we define $Eu := \lim_{m\to\infty} Eu_m$. □

Definition 1.71. We call Eu an *extension* of u to $\mathbb{R}^n$.

1.10.2 Traces

[Evans, Sec 5.5] Goal: Heuristically, we want to give meaning to "restricting" some Sobolev function to ∂U.

Theorem 1.72 (Evans, Thm. 5.5.1)**.** *Let $1 \leq p < \infty$. Let $U \subset \mathbb{R}^n$ be open with C^1-boundary. Then, there exists a bounded linear operator $T : W^{1,p}(U) \to L^p(\partial U)$ such that:*

1. *$Tu = u|_{\partial U}$ if $u \in W^{1,p} \cap C(\overline{U})$.*
2. *$\|Tu\|_{L^p(\partial U)} \leq C_{p,U}\|u\|_{W^{1,p}(U)}$ for all $u \in W^{1,p}(U)$.*

Proof. Step 1: Assume first that $u \in C^1(\overline{U})$ and suppose $x^0 \in \partial U$ and suppose that ∂U is flat near that point, ie, $\{x_n = 0\}$. Place two balls at x^0, B with with radius r and $\hat{B}$ with radius $r/2$. As usually, let $B^+ = B \cap \{x_n \geq 0\}$. Denote by Γ that portion of ∂U within $\hat{B}$. Select $\zeta \in C_c^\infty(B)$ with $\zeta \geq 0$ in B, $\zeta \equiv 1$ on $\hat{B}$. Let $x' := (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} \equiv \{x_n = 0\}$. Then:

$$\int_\Gamma |u(x',0)|^p \, \mathrm{d}x' \leq \int_{\{x_n=0\}} \zeta(x',0)|u(x',0)|^p \, \mathrm{d}x' = -\int_{B^+} \partial_{x_n}(\zeta(x',x)n)|u(x',x_n)|^p) \, \mathrm{d}x' \, \mathrm{d}x_n$$

$$= -\int (\partial_{x_n}\zeta)|u|^p + p|u|^{p-1}(sign(u))(\partial_{x_n}u)\zeta) \, \mathrm{d}x' \, \mathrm{d}y \leq C\int_{B^+}(|\partial_{x_n}\zeta||u|^p + |u|^{p-1}|\partial_{x_n}u||\zeta|) \, \mathrm{d}x' \, \mathrm{d}x_n \leq \tilde{C}\int_{B^+}(|u|^p + |\nabla u|^p) \, \mathrm{d}x' \, \mathrm{d}x_n.$$

Here we used:

$$|u|^{p-1}|\nabla u| \leq C(|u|^p + |\nabla u|^p)$$

via Young's Inequality:

$$ab \leq \frac{a^q}{q} + \frac{b^r}{r}; \quad \frac{1}{q} + \frac{1}{r} = 1.$$

so

$$a^{p-1}b \leq \frac{a^p}{q} + \frac{b^p}{p}; \quad q = \frac{p}{p-1}.$$

Step 2: For general ∂U, straighten out as before ∂U near $x^0 \in \partial U$ and proceed as in Step 1. Then by change of variables:

$$\int_\Gamma |u|^p \, \mathrm{d}s \leq C\int_U |u|^p + |\nabla u|^p \, \mathrm{d}x$$

for $x^0 \in \Gamma$ some open subset of ∂U.

Step 3: Since ∂U is compact, there exists finitely many $x^{0,i} \in \partial U$ and open subsets $\Gamma_i \subset \partial U$ such that $\partial U = \cup_1^n \Gamma_i$ and $\|u\|_{L^p(\Gamma_i)} \leq C_\Gamma \|u\|_) W^{1,p}(U)$. Thus if we set for $u \in C^1(U)$, $Tu = u|_{\partial U}$ then

$$\|Tu\|_{L^p(U)} \leq C\|u\|_{W^{1,p}(U)}.$$

Step 4: For arbitrary $u \in W^{1,p}(U)$, take $\{u_m\} \subset C^\infty(\overline{U})$ such that $\|u_m - u\|_{W^{1,p}(U)} \to 0$. Then define $Tu := \lim_{m\to\infty} Tu_m$ in $L^p(\partial U)$. □

Q: What's the kernel of T?

Theorem 1.73. *Let $1 \leq p < \infty$ and let $U \subset \mathbb{R}^n$ be open and bounded with C^1-boundary. Suppose $u \in W^{1,p}(U)$. Then $Tu = 0$ iff $u \in W_0^{1,p}(U)$. (Recall $W_0^{1,p}(U) := \overline{C_c^\infty(U)}^{\|\cdot\|_{W^{1,p}(U)}}$.*

Proof. Let $u \in W_0^{1,p}(U)$. Then, there is some $\{u_m\} \subset C_c^\infty(U)$ such that $u_m \to u$. Then, $Tu := \lim_n Tu_m = 0$.

Now suppose $Tu = 0$ on ∂U. Construct an approximating sequence of $\{u_m\}_m \subset C_c^\infty(U)$ such that $\|u - u_m\|_W \to 0$. Using partition of unity and local flattening of ∂U, we may assume: $u \in W^{1,p}(\mathbb{R}^n)$ where u has support in $\overline{\mathbb{R}^+}$ and $Tu = 0$ on $\partial\mathbb{R}_+^n \equiv \mathbb{R}^{n-1}$.

Since $Tu = 0$ on $\mathbb{R}^{n-1}$, there exists $\{u_m\}_m \subset C^1(\overline{\mathbb{R}^n}_+)$ such that $\|u_m - u\|_W \to 0$ and $Tu_m = u_m|_{\mathbb{R}^{n-1}} \to 0$ in $L^p(\mathbb{R}^{n-1})$. A technical estimate: For a.e. $x_n > 0$,

$$\int_{\mathbb{R}^{n-1}} |u(x',x_n)|^p \, \mathrm{d}x' \leq Cx_n^{p-1}\int_0^{x_n}\int_{\mathbb{R}^{n-1}} |\nabla(x',y)|^p \, \mathrm{d}x' \, \mathrm{d}y.$$

Next, let $\zeta \in C^\infty(\mathbb{R}_+)$ such that $\zeta \equiv 1$ on $[0,1]$ and $\zeta \equiv 0$ on $\mathbb{R}_+ \setminus [0,1]$, $0 \leq \zeta \leq 1$ and write $\zeta_m(x) := \zeta(xx_n)$, $x \in \mathbb{R}_+^n$ $m \in \mathbb{N}$. Write $w_m(x) := u(x)(1 - \zeta_m(x))$.

Now we show $\|w_m - u\|_{W^{1,p}(U)} \to 0$. □

1.10.3 Sobolev Inequalities

We only consider for $W^{1,p}(U)$.

Question: In what sense is a function in $W^{1,p}(U)$ " better " than just function in $L^p(U)$.

Ex: Let $U = B(0,1) \subset \mathbb{R}^n$ $n \geq 2$. Consider $f(x) := |x|^{-\gamma}$, $\gamma \in \mathbb{R}$. Then, on HW 2, $\partial_j f(x) = -\gamma\frac{x_j}{|x|}|x|^{-\gamma-1}$. Then, $f \in W^{1,p}(U)$ provided $\gamma < \frac{n}{p} - 1$:

$$\int_0^1 (r^{-\gamma-1})^p r^{n-1} \, \mathrm{d}r = r^{-(\gamma+1)p+n-1}.$$

So we need $-(\gamma+1)p + n - 1 > -1$. Since $f \in L^q(U)$ provides $\gamma < n/q$. The critical case comes when $n/p - 1 = n/q$ or when $q = \frac{np}{n-p} > p$, provided $p < n$. So in this sense, we get our function into higher L^q spaces.

We'll distinguish three regimes:

1. $1 \leq p < n$
2. $n < p \leq \infty$
3. $p = n$

For the first case, we expect that $W^{1,p}$ gives better L^q integrability. In the second case, we have the case $|x|^\alpha$, $\alpha > 0$. We get some improved regularity: morally better than just being continuous.

Case 1: $1 \leq p < n$.

Theorem 1.74 (Gagliardo-Nirenberg-Sobolev)**.** *Assume $1 \leq p < n$. There exists a $C(n,p) \equiv C$ such that for all $u \in C_c^1(\mathbb{R}^n)$*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

where $p^* = \frac{np}{n-p}$.

Remark 1.75 (Scaling Analysis)**.** Main observation: Both the left hand side and right hand side behave in a simple way under the following scaling transformation:

$$u(x) \mapsto u_\lambda(x) := u(\frac{x}{\lambda}).$$

Compute:

$$\|u_\lambda\|_{L^{P^*}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |u(\frac{x}{\lambda})|^{p^*} \, \mathrm{d}x\right)^{1/p^*} = \left(\int_{\mathbb{R}^n} |u(y)|^{p^*} \lambda^n \, \mathrm{d}y\right)^{1/p^*} = \lambda^{n/p^*}\|u\|^{L^{p^*}}(\mathbb{R}^n)$$

A similar argument with change rule gives:

$$\|\nabla u_\lambda\|_{L^p(\mathbb{R}^n)} = \lambda^{-1}\lambda^{n/p}\|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

So if theorem is true, then the bound must hold uniformly for all U_λ, $\lambda > 0$, ie:

$$\lambda^{n/p^*}\lambda^1\lambda^{-n/p} \leq \frac{C\|\nabla u\|_{L^p(\mathbb{R}^n)}}{\|u\|_{L^{p^*}(\mathbb{R}^n)}}.$$

So to avoid contradiction, we need that $n/p^* + 1 - n/p = 0$, or that $1/p^* = 1/n(n/p - 1) = \frac{n-p}{np}$.

Lemma 1.76 (Loomis-Whitney)**.** *For each $j = 1, \ldots, n$, let f_j be a nonnegative function measurable of all $x_1, \ldots, x_n$ except for x_j. Then:*

$$\int \cdots \int f_1 \ldots f_n \, \mathrm{d}x_1 \ldots \, \mathrm{d}x_n \leq \|f_1\|_{L^{n-1}(\mathbb{R}^{n-1})} \ldots \|f_n\|_{L^{n-1}(\mathbb{R}^{n-1})}.$$

Proof. Start by integrating $f_1 \ldots f_n$ in x_1:

$$\int_{\mathbb{R}} f_1 \ldots f_n \, \mathrm{d}x_1.$$

Since f_1 does not depend on x_1, we get:

$$f_1 \int_{\mathbb{R}} f_2 \ldots f_n \, \mathrm{d}x_1 \leq f_1\|f_2\|_{L{x_1}^{n-1}(\mathbb{R})} \ldots \|f_n\|_{L^{n-1}_{x_n}(\mathbb{R})}.$$

(This is Holder's Inequality + induction.) Next we integrate in x_2, using the fact that $\|f_2\|_{L_{x_2}(\mathbb{R})}$ is independent of x_2. (Note that $\frac{1}{n-1} + \ldots \frac{1}{n-1}$.) Then:

$$\int\int f_1 \ldots f_n \, \mathrm{d}x_1 \, \mathrm{d}x_2 \leq_{\text{Holder in x2}} \|f_2\|_{L_{x_1^{n-1}}(\mathbb{R})}\|f_1\|_{L^{n-1}_{x_2}(\mathbb{R}^2)}\|f_3\|_{L^{n-1}_{x_2}(\mathbb{R})} \cdots \|f_n\|_{L^{n-1}_{x_1,x_2}(\mathbb{R}^2)}.$$

Keep going. We achieve the result. ∎

Theorem 1.77. *Let $n \geq 2$ and $u \in C_c^1(\mathbb{R}^N)$. Then:*

$$\|u\|_{L^{n/(n-1)}(\mathbb{R}^n)} \leq \|\nabla u\|_{L^1(\mathbb{R}^n)}.$$

Proof. For $j = 1, \dots n$:

$$|u(x)| = |u(x_1, \dots, x_j, \dots, x_n)| = \int_{-\infty}^{x_j} \partial_j u(x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_n) \, \mathrm{d}x \le \int_{\mathbb{R}} |\nabla u(x_1, \dots x_j, y, x_{j+1}, \dots, x_n)| \, \mathrm{d}y.$$

Set: $g_j(x) := (\int_{\mathbb{R}} |\nabla u(x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_n)| \, \mathrm{d}y)^{1/(n-1)}$
Thus:

$$|u(x)|^{n/(n-1)} \le \Pi_{j=1}^n g_j(x).$$

Thus by Loomis-Whitney,

$$\int_{\mathbb{R}^n} |u(x)|^{n/(n-1)} \, \mathrm{d}x \le \Pi_{j=1}^n \|g_j\|_{L^{n-1}(\mathbb{R}^{n-1})} = \|\nabla u\|_{L^1(\mathbb{R}^n)}^{n/(n-1)}$$

Observe that:

$$\|g_j\|_{L^{n-1}(\mathbb{R}^{n-1}} = \|\nabla u\|_{L^1}^{1/(n-1)}$$

. ◻

Proof. Proof of theorem Building blocks:

- Loomis-Whitney inequality
- $W^{1,1}(\mathbb{R}^n)$ GNS inequality.

For $u \in C_c^1(\mathbb{R}^n)$ and $1 < p < n$:

$$\int_{\mathbb{R}^n} |u(x)|^{p^*} \, \mathrm{d}x = \int_{\mathbb{R}^n} ||u(x)|^\gamma|^{n/(n-1)} \, \mathrm{d}x.$$

where $\gamma = p^* \frac{n-1}{n}$. We apply $W^{1,1}$-GNS. We get:

$$\le \int_{\mathbb{R}^n} (|\nabla(|u(x)|^\gamma) \, \mathrm{d}x)^{n/(n-1)} \le \gamma^{n/(n-1)} \left(\int_{\mathbb{R}^n} |u(x)|^{\gamma-1} |\nabla u(x)| \, \mathrm{d}x \right)^{n/(n-1)} \le \|\nabla u\|_{L^p} \|u^{\gamma-1}\|_{L^{p/(p-1)}}$$

But now here we have, by Holder:

$$\int_{\mathbb{R}^n} |u(x)|^{\gamma-1} |\nabla u(x)| \, \mathrm{d}x \le \||u|^{\gamma-1}\|_{L^{p/(p-1)}(\mathbb{R}^n)} \|\nabla u\|_{L^p(\mathbb{R}^n)}$$

Therefore, we have that:

$$\|u(x)\|_{L^{p^*}}^{p^*} \le \gamma^{n/(n-1)} \||u|^{\gamma-1}\|_{L^{p/(p-1)}(\mathbb{R}^n)} \|\nabla u\|_{L^p}^{n/(n-1)}$$

Now, we also have that:

$$\||u|^{\gamma-1}\|_{L^{p/(p-1)}}^{n/(n-1)} = \|u\|_{L^{p/(p-1)(\gamma-1)}}^{(\gamma-1)n/(n-1)}$$

We find:

$$(\gamma - 1)\frac{n}{n-1} = p^* - \frac{n}{n-1} \implies \frac{p}{p-1}(\gamma - 1) = p^*.$$

Thus:

$$\|u\|_{L^{p^*}(\mathbb{R}^n)}^{p^*} \le \gamma^{n/(n-1)} \|u\|_{L^{p^*}}^{p^* - n/(n-1)} \|\nabla u\|_{L^p}^{n/(n-1)}$$

Thus:

$$\|u\|_{L^{p^*}}^{n/(n-1)} \le \gamma^{n/(n-1)} \|\nabla u\|_{L^p}^{n/(n-1)}.$$

Now we take exponents, and this completes the proof.

◻

Theorem 1.78 (Sobolev Inequalities for $W^{1,p}(U)$ for $1 \le p < n$)**.** *Let $U \subset \mathbb{R}^n$ be open and bounded with C^1-boundary. Then $u \in L^{p^*}(U)$ with*

$$\|u\|_{L^{p^*}(U)} \le C\|u\|_{W^{1,p}(U)}$$

Proof. Since ∂U is C^1, by extension theorem there exists $\bar{u} = Eu$ in $W^{1,p}(\mathbb{R}^n)$ such that $\bar{u} = u$ in U a.e., $\bar{u}$ has compact support, and $\|\bar{u}\|^{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(U)}$. By mollification, there exists $\{u_m\}_m \subset C_c^\infty(\mathbb{R}^n)$ such that $\|u_m - \bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \to 0$. GNS, $W^{1,p}$ version, for all $l, m \geq 1$:

$$\|u_m - u_l\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|\nabla u_m - \nabla u_l\|_{L^p} \to 0.$$

Thus, $u_m \to \bar{u}$ in L^{p^*} as well. Since by $W^{1,p}$ GNS, we have for all m,

$$\|u_m\|_{L^{p^*}} \leq C\|\nabla u_m\|_{L^p}.$$

we get

$$\|\bar{u}\|_{L^{p^*}} \leq C\|\nabla \bar{u}\|_{L^p}.$$

Thus:

$$\|u\|_{L^{p^*}(U)} \leq \|\bar{u}\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|\nabla \bar{u}\|_{L^p(\mathbb{R}^n)} \leq \tilde{C}\|u\|_{W^{1,p}(U)}.$$

□

Now we look at case 2: $p > n$.

Definition 1.79. Assume $U \subset \mathbb{R}^n$ open.

- If $u$is bounded and continuous, we denote:
$$\|u\|_{C(\bar{U}} := \sup_{x \in U} |u(x)|.$$
- The Holder semi-norm of regularity index, $0 < \gamma \leq 1$ for $u : U \to \mathbb{R}$ is:
$$[u]_{C^{0,\gamma}}(\bar{U}) := \sup_{x,y \in U; x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\gamma}.$$
- And the $C^{0,\gamma}(\bar{U})$ norm is:
$$\|u\|_{C^{0,\gamma}(\bar{U})} := \|u\|_{C(\bar{U})} + [u]_{C^{0,\gamma}(\bar(U)}.$$

Theorem 1.80 (Morrey's Inequality)**.** *Assume $n < p \leq \infty$. Let $\gamma = 1 - \frac{n}{p}$. Then there exists $C \equiv C(n,p) < 0$ such that for all $u \in C^\infty(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$,*

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

Lemma 1.81. *Let $u \in C^1(\overline{B(x,r)})$ for some $x \in \mathbb{R}^n$. Then*

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |u(y) - u(x)| \,\mathrm{d}y \leq \frac{1}{|S^{n-1}|}\int_{B(x,r)} \frac{|\nabla u(y)|}{|x-y|^{n-1}} \,\mathrm{d}y.$$

Proof. Write:

$$\int_B |u(x) - u(y)| \,\mathrm{d}y = \int_0^r \int_{\partial B(x,r')} |u(y) - u(x)| \,\mathrm{d}S(y) \,\mathrm{d}r'.$$

Using FTC:

$$\int_{\partial B(x,r')} |u(y) - u(x)| \,\mathrm{d}S(y) = \int_{\partial B(0,1)} |\underbrace{u(x + r'z) - u(x)}_{\int_0^{r'} \frac{\mathrm{d}}{\mathrm{d}s}(u(x+sz)}| \,\mathrm{d}S(z) \leq (r')^{n-1}\int_{\partial B(0,1)}\int_0^{r'} s^{-(n-1)}|\nabla(x+sz)|s^{n-1} \,\mathrm{d}s \,\mathrm{d}S(z) = (r')^{n-1}\int_{B(}$$

Here we used $y = x + sz$, with $|z| = 1$, so $|y - x| = s$. Thus:

$$\int_{B(x,r)} |u(y) - u(x)| \,\mathrm{d}y \leq \int_0^r (r')^{n-1}\int_{B(x,r')} \frac{|\nabla u(y)|}{|x-y|^{n-1}} \,\mathrm{d}y \,\mathrm{d}r' \leq \frac{r^n}{n}\int_{B(x,r)} \frac{|\nabla u(y)|}{|x-y|^{n-1}} \,\mathrm{d}y$$

Then use $\frac{1}{|B|}\frac{r^n}{n} = \frac{1}{|S^{n-1}|}$.

□

Proof of Morrey's inequality. First we bound

$$\|u\|_{C^0(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} |u(x)|.$$

For any $x \in \mathbb{R}^n$ and $r > 0$ to be chosen:

$$|u(x)| \leq |u(x) - u(y)| + |u(y)|.$$

Now average over $B(x,r)$:

$$|u(x)| \leq \frac{1}{|B(x,r)|}\int_{B(x,r)} |u(y) - u(x)\ \mathrm{d}y + \frac{1}{|B(x,r)|}\underbrace{\int_{B(x,r)} |u(y)|\ \mathrm{d}y}_{\leq \frac{1}{|B(r,x)|}|B(x,r)|^{(p-1)/p}\|u\|_{L^p(B(x,r)}\leq Cr^{-n/p}\|u\|_{L^p(\mathbb{R}^n)}}.$$

Now we look a the first term:

$$(1) \leq C\int_{B(x,r)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\ \mathrm{d}y \leq C\left(\int_{B(x,r)}\left(\frac{1}{|x-y|}\right)^{p/(p-1)}\ \mathrm{d}y\right)^{(p-1)/p}\|\nabla u\|_{L^p(B(x,r))} \leq C\left(\int_0^r s^{-(n-1)p/(p-1)}s^{n-1}\ \mathrm{d}s\right)^{(p-1)/p} \leq C$$

So we get:

$$\leq Cr^\gamma\|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

Now pick $r = 1$, and we get that:

$$\sup_{x\in\mathbb{R}^n}|u(x)| \leq C(\|u\|_{L^p} + \|\nabla u\|_{L^p}).$$

Next we estimate the Holder semi-norm: For any $x, y \in \mathbb{R}^n$, $x \neq y$ with $|x-y| = r$,

$$\frac{1}{|B(x,r)\cap B(y,r)|}|u(x)-u(y)| \leq \int_{B(x,r)\cap B(y,r)} |u(x)-u(z)|+|u(z)-u(y)|\ \mathrm{d}z \leq C\left(\int_{B(x,r)}|u(x)-u(y)|\ \mathrm{d}z + \frac{1}{|B(y,r)|}\int_{B(y,r)}|u(y) - u\right.$$

Using the fact that $|B(x,r)\cap B(y,r)| \sim r^n \sim ||B(x,r)|$.

$$\leq C\int_{B(x,r)}\frac{|\nabla u(z)|}{|x-z|^{n-1}} + C\int_{B(y,r)}\frac{|\nabla u(z)|}{|y-z|^{n-1}}\ \mathrm{d}z \leq Cr^\gamma\|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

□

Definition 1.82. We say u^* is a version of a given function u provided $u = u^*$ a.e.

By approximation and extension

Theorem 1.83 (Sobolev inequality)**.** *Let $U \subset \mathbb{R}^n$ be open and bounded with C^1-boundary and assume $\infty \geq p > n$ and $u \in W^{1,p}(U)$. Then u has a version $u^* \in C^{0,\gamma}(\overline{U})$ for $\gamma = 1 - n/p$ with the bound:*

$$\|u^*\|_{C^{0,\gamma}(\overline{U}} \leq C\|u\|_{W^{1,p}(U)}.$$

In the case when $p = n$, $W^{1,p}(\mathbb{R}^n)$ does not embed into $L^\infty(\mathbb{R}^n)$ when $n \geq 2$.

Definition 1.84. Let $u \in L^1_{loc}$, the bounded mean oscilation semi-norm is defined as:

$$[u]_{BMO} := \sup_{x\in\mathbb{R}^n,\ r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|u(y) - (u)_{x,r}|\ \mathrm{d}y$$

with

$$(u)_{x,r} := \frac{1}{|B(x,r)|}\int_{B(x,r)} u(z)\ \mathrm{d}z.$$

Proposition 1.85. Let $u \in W^{1,p}(\mathbb{R}^n)$ for $n \geq 2$. Then:

$$[u]_{BMO(\mathbb{R}^n)} \leq C\|u\|_{W^{1,n}(\mathbb{R}^n)}.$$

Proof. Let $B(x,r)$ be any ball in $\mathbb{R}^n$. Then Poincare's inequality for a ball (with $p=1$) gives:

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |u-(u)_{x,r}|\ \mathrm{d}y \le Cr\frac{1}{|B(x,r)|}\int_{B(x,r)} |\nabla u(y)|\ \mathrm{d}y.$$

Then:

$$\le C\underbrace{\frac{1}{|B(x,r)|}|B(x,r)|^{(n-1)/n}}_{1}\|\nabla u\|_{L^p(B(x,r))}.$$

□

1.10.4 Compactness results

Recall: For $U\subset\mathbb{R}^n$ open and C^1-boundary, we have that $W^{1,p}\hookrightarrow L^{p^*}(U)$ for $1\le p<n$, is:

$$\|u\|_{L^{p^*}}\le C\|u\|_{W^{1,p}(U)};\quad p^*:=\frac{np}{n-p}$$

This also holds for all $1\le q<p^*$ by Holders. (This is known as a non-sharp Sobolev inequality.)

Goal: Show that $W^{1,p}(U)\hookrightarrow L^q(U)$ is a compact embedding for $1\le q<p^*$.

Definition 1.86. Let X,Y be Banach spaces, $X\subset Y$. Then, we say that X is compactly embedded into Y, written $X\subset\subset Y$ provided $\|x\|_Y\le C\|x\|_X$ for some $C>0$ and each bounded sequence in X is pre-compact in Y, it has a convergence sequence in Y.

Theorem 1.87 (Rellich -Kondrachov Compactness theorem)**.** *Assume $U\subset\mathbb{R}^n$ is open, bounded, with C^1-boundary. Suppose $1\le p<n$. Then, $W^{1,p}(U)\subset\subset L^q(U)$ for each $1\le q<p^*$.*

For a counter example: $q=p^*$: Let $U=B(0,1)\subset\mathbb{R}^n$. Take $u\in W^{1,p}(U)$ with $\mathrm{supp}(U)\subset U$ with $u\|_{L^{p^*}}=1$. Set $v_\epsilon(x):=\epsilon^{-n/p^*}u(x/\epsilon)$ for $0<\epsilon<1$. Can check:

- $\|v_\epsilon\|_{W^{1,p}(U)}\le C\|u\|_{W^{1,p}(U)}$ uniformly for all $0<\epsilon<1$.
- However $\|v_\epsilon\|_{L^{p^*}}=\|u\|_{L^{p^*}}(U)=1$.
- On the other hand $v_\epsilon\to 0$ pw. a.e. as $\epsilon\to 0$.
- So suppose there exists some $v_{\epsilon_n}\to v$ in L^{p^*}. And so $v_{\epsilon_n}\to 0$. But then we contract the norm being 1.
- "Enemies" to pre-compactness in $L^q(U)$
  - Scaling: ruled out by non-sharp embedding
  - Translation: ruled out by boundness of domain
  - Oscillation; example: $v_n(x):=e^{inx};\quad x\in(0,1)$. Then $v_n\to 0$ weakly in L^q: This is ruled out by boundedness in $W^{1,p}(U)$.

Recall:

Theorem 1.88 (Arzela-Ascoli)**.** *Let $K\subset\mathbb{R}^n$ be compact, let $\{f_k\}_k$ be a sequence of continuous functions on K such that*

- $\sup_k\|f_k\|_\infty<\infty$
- *For all $\epsilon>0$, there exists a $\delta>0$such that for all k, $|u_k(x)-u)_k(y)|\le\epsilon$ when $|x-y|\le\delta$.*

Then there exists a subsequence $\{f_{k_n}\}$ that converges uniformly on K.

Proof of theorem. Strategy: take $\{u_m\}_m\subset W^{1,p}(U)$. Smooth out to $u_m^\epsilon\in C^\infty(\overline{U})$. Apply Arzela-Ascoli to u_m^ϵ and control $\|u_m-u_m^\epsilon\|_{L^q}$ uniformly in m, then do a standard diagonal argument.

Fix $1\le q<p^*$. Let $\{u_m\}$ be a bounded sequence in $W^{1,p}(U)$.

Step 1: By Holder's and U being bounded, for $u\in W^{1,p}(U)$,

$$\|u\|_{L^q(U)}\le C\|u\|_{L^{p^*}}\le\tilde{C}\|u\|_{W^{1,p}(U)}$$

whence $W^{1,p}(U)\subset Ł^q(U)$.

Step 2: By extension theorem, wlop, we may assume $U = \mathbb{R}^n$ and all functions $\{u_m\}_m \subset W^{1,p}(\mathbb{R}^n)$ have compact support in some fixed bounded open set $V \subset \mathbb{R}^n$ such that $\sup_m \|u_m\|_{W^{1,p}(V} =: \tilde{C} < \infty$.

Step 3: Define $u_m^\epsilon := \eta_\epsilon u_m$, $\epsilon > 0$, $m \in \mathbb{N}$ and we may assume that all u_m^ϵ have support in V as well.

Step 4: Claim: $u_m^\epsilon \to u_m$ in $L^q(V)$ uniformly in m as $\epsilon \to 0$. If u_m is smooth, then $u_m^\epsilon - u_m(x) = \int_{B(0,1)} \eta(y)(u_m(x - \epsilon y) - u_m(x)\ \mathrm{d}y = \int_{B(0,1)} \int_0^1 (\nabla u_m)(x - \epsilon t y) \cdots y\ \mathrm{d}t\ \mathrm{d}y$. Thus:

$$\int_V |u_m^\epsilon(x) - u_m(x)|\ \mathrm{d}x \leq \epsilon \int_{B(0,1)} \eta(y) \int_0^1 \int_V |\nabla u_m(x - \epsilon t y)\ \mathrm{d}x\ \mathrm{d}t\ \mathrm{d}y \leq \epsilon \int_v |\nabla u_m(x)|\ \mathrm{d}z \leq \epsilon C \|\nabla u_m\|_{L^p(V)} \leq \epsilon C \tilde{C}.$$

Thus, $\|u_m^\epsilon - u_m\|_{L^1(V)} \to 0$ uniformly in m. Then, since $1 \leq q < p^*$, using the interpolation inequality

$$\|u_m^\epsilon - u_m\|_{L^q(V)} \leq \|u_m^\epsilon - u_m\|_{L^1(V)}^\theta \|u_m^\epsilon - u_m\|_{L^{p^*}(V)}^{1-\theta)}$$

with $\frac{1}{q} = \frac{\theta}{1} + \frac{1-\theta}{p^*}$. Can check with Holder. We thus infer, using the uniform boundedness in L^{p^*}.

$$\|u_m^\epsilon - u_m\|_{L^q(V)} \leq C \|u_m^\epsilon - u_m\|_{L^1}^\theta.$$

And so we achieve uniform convergence.

Step 5: Now we bring in Arzela-Ascoli. Claim: For each fixed $\epsilon > 0$, the sequence $\{u_m^\epsilon\}_m$ is uniformly bounded and equicontinuous. This follows by: if $x \in \mathbb{R}^n$, for all $m \in \mathbb{N}$:

$$|u_m^\epsilon(x)| \leq \int_{B(x,\epsilon)} \eta_\epsilon(x - y)|u_m(y)|\ \mathrm{d} \leq \|\eta_\epsilon\|_\infty \|u_m\|_{L^1(V)} \leq \epsilon^{-n} C \tilde{C}.$$

This bound is uniform. And:

$$|\nabla u_m^\epsilon(x)| \leq \|\nabla \eta_\epsilon\|_\infty \|u_m\|_{L^1} \leq \frac{C\tilde{C}}{\epsilon^{n+1}}.$$

Using FTC, this implies that we have equicontinuity.

Step 6: Now fix $\delta > 0$. We want to show that there exists a subsequence $\{u_{m_j}\}_j$ such that $\lim_{j,k\to\infty} \sup \|u_{m_j} - u_{m_k}\|_{L^q}(V) \leq \delta$. We look at: $u_{m_j} - u_{m_k} = (u_{m_j} - u_{m_j}^\epsilon) + (u_{m_j}^\epsilon - u_{m_k}^\epsilon) + (u_{m_k}^\epsilon - u_{m_k})$.

Use uniform smooth approximation to selection $\epsilon > 0$ so small so that for all m:

$$\|u_m^\epsilon - u_m\|_{L^q(V)} \leq \delta/2.$$

Then, Use Arzela-Ascoli and step 5 to obtain a subsequence $\{u_{m_j}^\epsilon\}_j \subset \{u_m^\epsilon\}_m$ which converges uniformly in L^∞ on V and thus:

$$\lim_{j,k\to\infty} \sup \|u_{m_j^\epsilon - u_{m_k}^\epsilon}\|_{L^q(V)} = 0.$$

Thus,

$$\lim_{j,k\to\infty} \sup \|u_{m_j} - u_{m_k}\|_{L^q(V)} \leq \delta.$$

Step 7: Apply Diagonal argument for $\delta = 1, 1/2, \ldots$. This gives the subsequence.

□

1.11 Second-order linear elliptic-equations [Evans, Chapter 6; Brezis: FA, Sobolev spaces and PDEs, Chapters 8,9]

Goal: Existence, uniqueness, and regularity of weak solutions to the BVP:

$$Lu = f; \quad \in U$$

$$u = 0; \quad x \in \partial U$$

where $U \subset \mathbb{R}^n$ open and bounded and

$$Lu := -\sum_{i,j=1}^n \partial_j(a^{ij}(x)\partial_j u_+ \sum_{i=1}^n b^i(x)\partial_i u + c(x)u.$$

In what follows, we assume:

- Symmetry conditions $a^{ij} = a^{ji}$ for all i, j.
- a^{ij}, $b^i, c \in L^\infty(U)$
- L is uniformly elliptic, meaning:

Definition 1.89. We say that the linear partial differential operator L is uniformly elliptic if there exists some $\theta > 0$ such that
$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_j\xi_i \geq \theta|\xi|^2$$
for all $\xi \in \mathbb{R}^n$ and a.e. $x \in U$.

Remark 1.90. We say that the PDE $Lu = f$ is in divergence form if L is given by the above, and in non-divergence form provided L is given by:
$$Lu := -\sum_{i,j=1}^{n} a^{ij}\partial_i\partial_j u + \sum_{1}^{n} b^i(x)\partial_i u + c(x)u.$$

1.11.1 Weak Solutions

Heuristics: Suppose u is a smooth solution to $Lu = f$. Multiply by a test function ϕ. We integrate over the domain. We
$$\int_U \phi(-\sum_{i,j=1}^{n} \partial_j(a^{ij}(x)\partial_i u) + \sum_{i=1}^{n} b^i\partial_i u + cu) \, \mathrm{d}x = \int_U \phi f \, \mathrm{d}x.$$
$$\sum_{i,j=1}^{n}(-1)\int \phi\partial_j(a^{ij}\partial_i u) \, \mathrm{d}x = \sum_{i,j=1}^{n}\int_U \partial_j\phi(a^{ij})\partial_i u \, \mathrm{d}x.$$

Observe that we can make things weaker on the right hand side by letting $u \in H_0^1(U)$ and $\phi \in H_0^1(U)$.

Definition 1.91.

- The bilinear form $B : H_0^1(U) \times H_0^1(U) \to \mathbb{R}$ associated with the divergence form elliptic operator L is:
$$B[u, v] := \int_U (a^{ij}(\partial_i u)(\partial_j v) + b^i(\partial_i u)v + cuv) \, \mathrm{d}x.$$
- We say that $u \in H_0^1(U)$ is a weak solution to the BVP $Lu = f$ and $u = 0$ on ∂U with $f \in L^2(U)$ if $B[u, v] = (f, v)_{L^2}$ for all $v \in H_0^1(U)$.

Remark 1.92.

- The weak form is also called the variational form of the BVP.
- Check: u is a strong classical solution, then it is also a weak solution. Suitable assumptions may be necessary here. Furthermore, u is a classical solution to BVP iff $u \in C^2(U) \cap C(\overline{U})$ is a weak solution to BVP.

1.11.2 Existence of Weak Solutions

Theorem 1.93 (Lax-Milgram)**.** *Assume H is a real Hilbert space with norm $\|\cdot\|$ and inner product $(\cdot, \cdot)$. We let $\langle\cdot, \cdot\rangle$ denote the pairing of H with its dual space. Assume that $B : H \times H \to \mathbb{R}$ is a bilinear mapping for which there exists constants $\alpha, \beta > 0$ such that:*

1. *(Boundedness) $|B[u, v]| \leq \alpha\|u\|\|v\|$ for all $u, v \in H$*
2. *(Coercivity) $\beta\|u\|^2 \leq B[u, u]$ for all $u \in H$.*

Let $f : H \to \mathbb{R}$ be a bounded linear functional on H. Then there exists a unique weak $u \in H$ such that $B[u, v] = \langle f, v\rangle$ for all $v \in H$.

Energy estimates

Proposition 1.94. Under the preceding assumptions there exists constants $\alpha, \beta > 0$ and $\gamma \geq 0$ such that

- $|B[u,v]|\alpha\|u\|_{H_0^1}\|v\|_{H_0^1}$.
- $\beta\|u\|_{H_0^1}^2 \leq B[u,u] + \gamma\|u\|_{L^2}^2$.

Proof.
- By Holder's inequality $|B[u,v]| \leq \sum_{ij}\|a^{ij}\|_\infty\|\nabla u\|_{L^2}\|\nabla v\|_{L^2} + \sum_i\|b^i\|_\infty\|\nabla u\|_2\|v\|_2 + \|c\|_\infty\|u\|_2\|v\|_2 \leq \alpha\|u\|_{H_0^1}\|v\|_{H_0^1}$.
- By uniform ellipticity:

$$B[u,u]+\int_U\sum_i|b^i(x)||\partial_i u||u|\ \mathrm{d}x+\int_U|c(x)||u|^2\ \mathrm{d}x \geq B[u,u]-\int_U\sum_i b^i(x)(\partial_i u)u\ \mathrm{d}x-\int_U c(x)u^2\ \mathrm{d}x = \int_D\sum_{ij}a^{ij}(x)(\partial_i u)(\partial_j u)\ \mathrm{d}x \geq$$

Now use Youngs, with $\epsilon > 0$ small enough so small that $\sum_i\|b^i\|_\infty\epsilon \leq \theta/2$:

$$\|\nabla u\|_2\|u\|_2 \leq \epsilon\|\nabla u\|_2^2 + \frac{1}{4\epsilon}\|u\|_2^2.$$

We get:

$$\frac{\theta}{2}\int_U|\nabla u|^2\ \mathrm{d}x \leq B[u,u] + C\|u\|_2^2.$$

Finally, recall the Poincare-type estimate: there exists $C > 0$ such that $\|u\|_2 \leq C\|\nabla u\|_2$ for all $u \in H_0^1(U)$. After a Young's-type inequality, this gives the bound.

□

Referring to the previous proposition, observe that if $b^i \equiv 0$ and $c(x) \geq 0$. Then,

$$B[u,u] = \int_U\left(a^{ij}(x)(\partial_i u)(\partial_j u) + c(x)u^2\right)\ \mathrm{d}x \geq \theta\int_U|\nabla u|^2\ \mathrm{d}x.$$

and use that for $u \in H_0^1$, $\|u\|_2 \leq C\|\nabla u$
$_2$. Then we do the usual trick, and we get there. This would give us coercivity, and Lax-Milgram would readily apply.

To quickly recall Lax-Milgram: Assume H is a real Hilbert space with $\|\cdot\|$ and inner product $(\cdot,\cdot)$. We let $\langle\cdot,\cdot\rangle$ denote a duality pairing.

Theorem 1.95 (Lax-Milgram)**.** *Assume $B : H \times H \to \mathbb{R}$ is bounded and coercive, that is, there exists $\alpha, \beta > 0$ such that*

1. $|B[u,v]| \leq \alpha\|u\|_H\|v\|_H$
2. $\beta\|u\|_H^2 \leq B[u,u]$.

Let $f : H \to \mathbb{R}$ be a bounded linear functional on H. Then there exists a unique $u \in H$ such that $B[u,v] = \langle f, v\rangle$ for all $v \in H$.

Proof. Fix $u \in H$. Then $B[u,v]$ is a bounded linear functional on H. Now the Reisz represetion theorem says that there exists a unique $w_u \in H$ such that $B[u,v] = (w,v)_H$ for all $v \in H$. Now define $A : H \to H$ by $u \mapsto w_u$. Then A is linear and is bounded because the bilinear form is bounded: $\|Au\|^2 = \|w_u\|^2 = B[u,w_u] \leq \alpha\|u\|\|w_u\| = \alpha\|u\|\|Au\|$. Hence $\|Au\| \leq \alpha\|u\|$. View $B[u,v] = (Au,v)_H$.

Claim: A is bijective. Injectivity: For $u \in H$, $\beta\|u\|^2 \leq B[u,u] = (Au,u) \leq \|Au\|\|u\|$. Thus, $\beta\|u\| \leq \|Au\|$. So A is injective. Surjective: We argue that the range of A is closed. Let $Au_n \to z \in H$, so Au_n is Cauchy. But then by the earlier estimate, $\{u_n\}$ is Cauchy in H. Thus, $u_n \to u \in H$, and thus $Au_n \to Au$. Thus $Au = z$, and so $z \in range(A)$. Now suppose $range(A) \neq H$. Since the range is closed, there exists $v \neq 0 \in [range(A)]^\perp$. Thus, $B[v,v] = (Av,v) = 0$. By coercivity, this implies that $v = 0$, which contradictions the assumption. Thus A is bijective.

Now to prove the theorem, by Riesz Representation theorem, there exists a unique $w \in H$ such that $(w,v) = \langle f, v\rangle$. By the claim, since A bijective there exists some $u \in H$ such that $Au = w$. Thus, $B[u,v] = (Au,v) = (w,v) = \langle f, v\rangle$. □

Theorem 1.96 (First existence result for weak-solutions)**.** *Under the preceding assumptions, there exiss $\gamma \geq 0$ such that for each $\mu \geq \gamma$ and each function $f \in L^2$, there exists a unique weak solution $u \in H_0^1$ to the boundary value problem $Lu + \mu u = f$, and $u = 0$ on ∂U.*

Proof. Define $B_\mu[u,v] := B[u,v] + \mu(u,v)_2$. Then, B_μ satisfies the assumptions of Lax-Milgram, as $B_\mu[u,u] \geq \beta\|u\|_H^2 - \gamma\|u\|_2^2 + \mu\|u\|_2^2$. So for fixed $f \in L^2$, setting $\langle f, v\rangle := (f,v)_2$, we get a unique $u \in H_0^1$ such that $B_\mu[u,v] = \langle f, v\rangle$ for all $v \in H_0^1$. □

Characterization of the dual space of H_0^1:

Definition 1.97. We denote by $H^{-1}(U)$ the dual space to H_0^1, that is, it is the space of all bounded linear functionals on H_0^1.

Note, we do *not* identify the Hilbert space H_0^1 with H_0^1 itself. Rather, we think as follows:

- $H_0^1(U) \subset L^2(U) \simeq (L^2(U))^* \subset H^{-1}(U)$

Theorem 1.98.
- *Assume $f \in H^{-1}(U)$. There exist functions $f^0, f^1, \dots, f^n \in L^2$ such that $\langle f, v\rangle = \int_U (f^0 v + \sum_{j=1}^n f^i \partial_i v)\, \mathrm{d}x$ for any $v \in H_0^1$.*
- *Furthermore, $\|f\|_{H^{-1}} = \inf\{(\int_U \sum_{j=0}^n |f^j|^2\, \mathrm{d}x)^{1/2} : \quad f$ satisfies the above for $f^0, f^1, \dots, f^n \in L^2(U)\}$.*

Proof. Given $u, v \in H_0^1(U)$, we define inner produce $(u,v)_H = \int_U uv \sum_1^n (\partial_i u)(\partial_j v) + uv\, \mathrm{d}x$. Let $f \in H^{-1}$. Apply Riesz representation theorem to get existence of unique function $u \in H_0^1$ such that $(u,v)_H = \langle f, v\rangle$ for all $v \in H_0^1$. Then take $f^0 = u$ and $f^i = \partial_i u$. This proves the first item with the choice.

For the second, see Evans, last section in chapter 5. $\square$

Theorem 1.99 (Fredholm Alternative for compact operators)**.** *LetH be a real Hilbert space and let $K : H \to H$ be a compact operator, $\lambda \in \mathbb{R} - \{0\}$. Then either $(K - \lambda)u = f$ has a unique solution in H for all $f \in H$ or $n := \dim(\ker(K - \lambda)) \in \mathbb{N}$ and $n = \dim(\ker(K^* - \lambda))$ in which case $(K - \lambda)u = f$ has a unique solution iff $f \in [\ker(K^* - \lambda)]^\perp$.*

Linear Algebra analogy: $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. Then $Ax = b$ has a solution iff $\langle y, b\rangle = 0$ for all $y \in \ker(A^T)$.

Definition 1.100.
- The operator L^* which is formally the adjoint of L is $L^* v := -\sum_{i,j=1}^n \partial_j(a^{ij}\partial_j v) - \sum_{i=1}^n \partial_i(b^i v) + cv$ provided $b^i \in C^1(\overline{U})$ for $1 \le i \le n$.
- The adjoint bilinear form $B^* : H_0^1 \times H_0^1 \to \mathbb{R}$ is defined by:

$$B^*[v,u] := B[u,v].$$

- We say that $v \in H_0^1$ is a weak solution of the adjoint problem $L^* = f$, $v = 0$ on ∂D provided $B^*[v,u] = (f,u)_{L^2}$ for all $u \in H_0^1$.

Theorem 1.101 (Second existence theorem for weak solutions; Evans theorem 6.2.4)**.**
- *Either for each $f \in L^2(U)$ there exists a unique weak solution u to the BVP $Lu = f$ and $u = 0$ on ∂D. Call this (# 1). Or else there exiss a weak $u \neq 0$ to $Lu = 0$ in D and $u = 0$ on ∂D. Call this (# 2).*
- *In case (# 2), the dimension of the subspace $N \subset H_0^1$ of weak solutions to (# 2) is finite and equals the dimension of the subspace $N^* \subset H_0^1$ of weak solutions to $L^* u = 0$ in U and $v = 0$ on ∂D. Finally, in case (# 2) the nonhomogeneous BVP has a non-unique weak solution if and only it $(f,v)_{L^2} = 0$ for all $v \in N^*$.*

Proof. Step 1: Choose $\mu = \gamma$ from our energy estimates. Define a new bilinear form $B_\gamma[u,v] := B[u,v] + \gamma(u,v)_{L^2}$ (which formally corresponds to operator $L_\gamma u := Lu + \gamma u$. Note B_γ is coercive in the sense that $B_\gamma[u,u] \ge \alpha\|u\|^2_{H_0^1}$. Thus by the first existence theorem (or Lax-Milgram) for each function $g \in L^2$, there exists a unique function $u \in H_0^1$ such that (# 3) $B_\gamma[u,v] = (g,v)_{L^2}$ for all $v \in H_0^1$. Whenever (#3) holds, we write $u = L_\gamma^{-1} g$. View $L_\gamma^{-1} : L^2(U) \to H_0^1(U)$.

Step 2: Observe $u \in H_0^1$ is a weak solution to (# 1), i.e., $Lu = f$ $u = 0$ on ∂D iff $B[u,v] = (f,v)_{L^2}$ for all $v \in H_0^1$ iff $B[u,v] + \gamma(u,v)_{L^2} = (f,v)_{L^2} + \gamma(u,v)_{L^2}$ iff $B_\gamma[u,v] = (f + \gamma u, v)_{L^2}$. That is, iff, $u = L_\gamma^{-1}(\gamma u + f)$. We rewrite this equality as $u - Ku = h$ with $Ku := \gamma L_\gamma^{-1} u$ and $h := L_\gamma^{-1} f$.

Step 3: We claim that $K : L^2 \to L^2$ is a linear, bounded, compact operator. By coercivity of $B_\gamma[u,u]$ there exists some $\beta > 0$ such that $\beta\|u\|_H \le B_\gamma[u,u] = (g,u)_{L^2}$ where u is the unique weak solution to $L_\gamma u = g$. Then $(g,u)_{L^2} \le \|g\|_{L^2}\|u\|_H$. Thus, $\|u\|_H \ leq \frac{1}{\beta}\|g\|_L$. But $u = L_\gamma^{-1} g$. $Kg = \gamma L_\gamma^{-1} g = \gamma u$. Thus, $\|Kg\|_H = \|\gamma u\|_H \le \frac{\gamma}{\beta}\|g\|_L$. Thus for any $g \in L^2$, $\|Kg\|_L \le C\|g\|_L$. Thus, K is a bounded operator. Since by Rellich-Koundrachov, $H_0^1 \subset\subset L^2$ (compactly embedded), K is in fact a compact operator.

Step 4: Now apply Fredholm alternative to $I - K : L^2 \to L^2$. Then either: (α)for each $h \in L^2$ the equation $u - Ku = h$ has a unique solution $u \in L^2$ OR (β) the equation $u - Ku = 0$ has nonzero solutions in L^2. Now observe: in case α by unraveling the definition, there exists unique weak solution to $Lu = f$ $u = 0$ on ∂D. Similarly, in case (*beta*) we obtain the assertions of our statement by unraveling definitions and looking more carefully at what Fredholm alternative gives you. $\square$

Definition 1.102. We call Σ the real spectrum of the operator L.

Theorem 1.103 (Third existence theorem for weak solutions)**.**
- *There exists at most countable set $\Sigma \subset \mathbb{R}$ such that the BVP $Lu = \lambda u + f$ in D and $u = 0$ on ∂D has a unique solution for each $f \in L^2$ iff $\lambda \notin \Sigma$.*

- *If Σ is infinite, then $\Sigma = \{\lambda_k\}_1^\infty$ with $\{\lambda_k\}$ nondecreasing.*

Proof. Let $\gamma \geq 0$ be the constant from the energy estimates proposition. Wlog, $\gamma > 0$ and $\gamma > -\gamma$. By Fredholm alternative (2nd existence result) the BVP $Lu = \lambda u + f$ and $u = 0$, view as $(L - \lambda)u = f$ $u = 0$ has a unique weak solution for each $f \in L^2$ iff $u \equiv 0$ is the unique weak solution to the homogenous problem $(L - \lambda)u = 0$ and $u = 0$. This in turn is true iff $u \equiv 0$ is the only weak solution to $Lu + \gamma u = (\gamma + \lambda)u$ iff $u = (L_\gamma^{-1} + \lambda)u = \frac{\lambda+\gamma}{\gamma} Ku$ iff $Ku = \frac{\gamma}{\gamma+\lambda} u$. Thus, $u \equiv 0$ is the only weak solution of this problem iff $\gamma/(\gamma + \lambda)$ is not an eigenvalue of K. Now, form Riesz-Schauder theorem on spectrum of compact operators on Hilbert spaces the assertion of the theorem follows. □

1.11.3 Regularity of Weak Solutions

Q: Is a weak solution to the linear elliptic PDE $Lu = f$ actually smoother?

Theorem 1.104 (H^2 regularity result)**.** *Let $U \subset \mathbb{R}^n$ be open and bounded. Assume $a^{ij} \in C^1(U)$, $b^i, c \in L^\infty(U)$ and $f \in L^2$. Suppose $u \in H^1$ is a weak solution of $Lu = f$ in U where $Lu = -\partial_j(a^{ij}\delta_i u) + b^i \partial_i u + cu$. Then $u \in H^2_{loc}(U)$ and for each open compactly supported set $V \subset\subset U$, we have that $\|u\|_{H^2} \leq C\|f\|_{L^2} + \|u\|_{L^2}$ with C depending only on V, U and the coefficients.*

Proof. Denote $\delta_k^h u(x) + \frac{u(x+he_k)-u(x)}{h}$. Derive uniform bounds for small $h > 0$ using the structural algebraic assumption of ellipticity.

Control difference quotients $\|\delta_k^h \partial_i u\|_{L^2(V)}$ for small h, then invoke the next theorem.

1. Fix open $V \subset\subset U$ and choose open W such that $V \subset\subset W \subset\subset U$. Pick a smooth cut-off ζ s.t. $\zeta \equiv 1$ on V, $\zeta \equiv 0$ on $\mathbb{R}^n \setminus W$ and $0 \leq \zeta \leq 1$.

2. Since u is a weak solution to $Lu = f$, $B[u, v] = (f, v)_{L^2}$ for all $v \in H_0^1(U)$. Then:

$$\sum_{1 \leq i,j, \leq n} \int a^{ij}(\partial_i u)(\partial_j v) \, \mathrm{d}x = \int_U \hat{f} v \, \mathrm{d}x$$

   where $\hat{f} := f - \sum_{i=1}^n b^i \partial_i u - cu$. Call the left hand side A and right hand side B.

3. For $1 \leq k \leq n$, fixed, choose $v := -\delta_k^{-h}(\zeta^h \delta_k^h u) \in H_0^1(U)$, which is well-defined for h sufficiently small. Now, idea is to plug in v.

$$A = -\sum_{1 \leq i,j \leq n} \int_U a^{ij}(\partial_i u)\partial_j(\delta_k^{-h} \zeta^2 \delta_k^h u) \, \mathrm{d}x.$$

□

Interlude/ technical preparation.

Let $U \subset \mathbb{R}^n$ be open and bounded. The let $V \subset\subset U$. Let $u \in L^1_{loc}(U)$.

Definition 1.105.

- The k-th difference quotient of size $h \neq 0$ is

$$(\delta_h^k u)(x) := \frac{u(x_h e_k) - u(x)}{h}$$

  for $x \in V$ and $h \in \mathbb{R}$, $0 < |h| < dist(V, \partial U)$.

- $\delta^h u := (\delta_1^h u, \dots, \delta_n^h u$.

Theorem 1.106.

1. *Suppose $1 \leq p < \infty$ and $u \in W^{1,p}(U)$. Then for each $V \subset\subset U$,*

$$\|\delta^h u\|_{L^p(V)} \leq C\|\nabla u\|_{L^p(U)}$$

   for some constant $C \equiv C(V, U)$ and all $0 < |h| < \frac{1}{2} dist(V, \partial U)$

2. *Assume $1 < p < \infty$, $u \in L^p(V)$ and there exists a $C > 0$ such that*

$$\|\delta^h u\|_{L^2} \leq C$$

   for all $0 < |h| < \frac{1}{2} dist(V, \partial U)$. Then $u \in W^{1,p}(V)$ with $\|\nabla u\|_{L^p} \leq C$.

Proof. 1. Assume $1 \le p < \infty$ and that u is in fact smooth. For $x \in V$, $1 \le K \le n$ and $0 < |h| < \frac{1}{2} dist(V, \partial U)$,

$$u(x + he_k) - u(x) = \int_0^2 \frac{d}{dt}(u(x + the_k)) \text{ dt} = \int_0^1 (\partial_k u)(x + the_k)h \text{ dt}.$$

Thus:

$$\left|\left|\delta_k^h u(x)\right| \le \int_0^1 |\nabla u(x_t he_k)| \text{ dt}.$$

Then:

$$\|\delta_k^h u\|_{L^p} \le \| \int_0^1 |\nabla(x + the_k)| \text{ dt}\|_{L^p} \le \int_0^1 \|\nabla u(\cdot + the_k)\|_{L^p} \text{ dt} \le \|\nabla u\|_{L^p(U)}.$$

This was for smooth functions, and now pass to smooth approximation for $u \in W^{1,p}$.

2. Suppose we have for all sufficiently small $0 < |h| < \frac{1}{2} dist(V, \partial U)$ and

$$\|\delta^h u\|_{L^p(V)} \le C.$$

Fix $1 \le k \le n$, $\phi \in C_c^\infty(V)$, then for small h:

$$\int_V u(x) \frac{\phi(x + he_k) - \phi(x)}{h} \text{ dx} = -\int_D \frac{u(x) - u(x - he_k)}{h} \phi(x) \text{ dx}.$$

Thus

$$\int_V u(\delta_k^h \phi) \text{ dx} = -\int_V (\delta_k^h u)\phi \text{ dx}.$$

"Integration by parts" for difference quotients. Since $1 < p < \infty$, L^p is reflexive. From the assumption, we have that a sequence $\partial_k^{-h_l} u$ with $h_l \to 0$ is uniformly bounded in L^p. Thus, it has a weakly convergence subsequence, ie, there exists subsequence $h_{l_m} \to 0$ and $v_k \in L^p$ such that $\delta_k^{-h_{l_m}} u \to V_k$ weakly. That is, for all $w \in L^{p'}$, we have $\int_D w(\delta_k^{-h_{l_m}} \text{ dx} \to \int_V wv_k \text{ dx}$. We we used that fact that a bounded sequence has a weakly convergence subsequence. Thus, for any $\phi \in C_c^\infty(V)$

$$\int_V u\partial_k \phi \text{ dx} = \lim_{h_{l_m} \to 0} \int_V u\delta_k^{l_m} \phi \text{ dx} = \lim_{h_{l_m} \to 0} \int_V \delta_k^{-h_{l_m}} u\phi \text{ dx} = -\int_V v_k \phi \text{ dx}.$$

Thus, v_k is the ∂_k weak partial derivative of u.

□

Remark 1.107.
- We do not require $u \in H_0^1(U)$.
- Since $u \in H_{loc}^2(U)$ we can infer that $Lu = f$ a.e. in U. Indeed, $B[u, \phi] = (f, \phi)_{L^2}$ with real IBP, $B[u, \phi] = (Lu, \phi)_{L^2} = (f, \phi)_{L^2}$. Apply Fundamental Lemma of Variational Calculus.

Of interior regularity. Goal is to control difference quotients $\|\delta_k^h \partial_j u\|_{L^2(V)}$ uniformly for all sufficiently small $h \ne 0$. We then apply the previous theorem to put another set of derivatives on $\partial_j u$.

1) Fix open $V \subset\subset U$ and choose an open set $V \subset\subset W \subset\subset U$. Pick a smooth cutoff ζ such that $\zeta = 1$ on V and $\zeta = 0$ outside W.

2) $u \in H^1(U)$ is a weak solution to $Lu = f$: $B[u, v] = (f, v)_{L^2}$, where $B[u, v] = \sum_{ij} \int_D a^{ij}(\partial_j uu)(\partial_i v) \text{ dx} = \int_U \tilde{f} v \text{ dx}$ where $\tilde{f} = f - \sum_i b^i \partial_i u - cu$.

3) For $k \in \{1 : n\}$ choose $v = -\delta_k^{-h}(\zeta^2 \delta_k^h u) \in H_0^1$ which is well-defined for sufficinetly small $|h| > 0$. Now, the idea is the following while picking up some junk:

$$\int a^{ij} \partial_i u \partial_j (\delta_k^{-h}(\zeta^2 \delta_k^h u)) \text{ dx} \to \int \delta_k^h (a^{ij} \partial_i u)\zeta^2 \delta_k^h \partial_j u \text{ dx} \to \int a^{ij} a^{ij} (\delta_k^h \partial_i u)(\delta_k^h \partial_j u)\zeta^2 \text{ dx}(\ge \theta |\delta_k^h \nabla|^2) \to \ge \theta \int |\delta_k^h \nabla u|^2 \zeta^2 \text{ dx}.$$

4) Estimating A: plug v as chosen into variational equation for u.

$$\partial_j(\delta_k^h(\zeta^2 \delta_k^h)) = \delta_k^{-h}(\zeta^2 \delta_k^h u); \quad \delta_k^h(a^{ij} \partial_i u) = a^{ij,h}(\delta_k^h \partial_i u) + (\delta_k^h a^{ij})\partial_i u.$$

$$\delta_k^h(vw) = v^h(\delta_k^h w) + (\delta_k^h v)v.$$

$$A = -\sum_{ij}\int_U a^{ij}(\partial_i u)\partial_j(\delta_k^{-h}(\zeta^2\delta_k^h u))\ \mathrm{d}x \sum_{ij}\int_U \delta_k^h(a^{ij}\partial_i u)\partial_j(\zeta^2\delta_k^h u)\ \mathrm{d}x$$

Thus

$$A = \sum_{i,j}\int_U a^{ij,h}(\delta_k^h\partial_i u)(\delta_k^h\partial_j u)\ \mathrm{d}x + \sum_{i,j}\int_U a^{ij,h}(\delta_k^h\partial_i u)(\delta_k^h u)2\zeta\partial_j\zeta + (\delta_k Tha^{ij}(\partial_i u)(\delta_k^h\partial_j u)\zeta^2 + (\delta_k^h a^{ij})(\partial_i u)(\delta_k^h u)2\zeta(\partial_j\zeta)\ \mathrm{d}x$$

.

Call the first term A_1 and all the remainder A_2. By uniform ellipticity, $A_1 \geq \theta\int_U \zeta^2|\delta_k^h\nabla u|^2\ \mathrm{d}x$. So we have:

$$\int \theta|\delta_k^h\nabla u|^2\ \mathrm{d}x \leq A_1 \leq |A_2| + |\int \tilde{f}v\ \mathrm{d}x|.$$

Moreover,

$$|A_2| \leq C\int_U \zeta|\delta_k^h\nabla u||\delta_k^h u| + \zeta\nabla u||\delta_k^h\nabla u| + \zeta\nabla u||\delta_k^h u|\ \mathrm{d}x$$

.

Here, observe that $\|a^{ij,h}\|_\infty$ is fine. Here we are using the fact that a^{ij} are C^1, and so $\delta_k^h a^{ij}$ are uniformly bounded independent of h. Thus, the above constant C is independent of h. Then for any $\epsilon > 0$:

$$|A_2| \leq \epsilon\int \zeta^2|\delta_k^h\nabla u|^2\ \mathrm{d}x + \frac{1}{\epsilon}C\int |\delta_k^h u|^2 + |\nabla u|^2\ \mathrm{d}x$$

Hence, for $\epsilon = \theta/2$, we obtain:

$$\frac{\theta}{2}\int |\delta_k^h\nabla u|\ \mathrm{d}x \leq C\int \nabla u|^2\ \mathrm{d}x + |\int \tilde{f}v\ \mathrm{d}x|.$$

Now we address the last term:

5)

$$|B||\int_U \tilde{f}v\ \mathrm{d}x| = |\int_U (f - \sum_i b^i\partial_j u - cu)(\delta_k^{-h}(\zeta^2\delta_k^h u)|^2)\ \mathrm{d}x| \leq C\int |(f| + |\nabla u| + |u|)|v|\ \mathrm{d}x.$$

Moreover, $\int_U |v|^2\ \mathrm{d}x = \int |\delta_k^{-h}(\zeta^2\delta_k^h u)|^2\ \mathrm{d}x \leq C\int_U |\nabla(\zeta^2\delta_k^h u)|^2\ \mathrm{d}x \leq C\int \zeta^2|\nabla\partial_k^h u|^2 + |\partial_j\zeta|\zeta\zeta|\delta_k^h u|^2\ \mathrm{d}x$. Thus, for any $\epsilon > 0$, we get

$$|B| \leq \epsilon\int_U \zeta^2|\delta_k^h\nabla u|\ \mathrm{d}x + \frac{1}{\epsilon}C\int_U f^2 + u^2 + \nabla u|^2\ \mathrm{d}x.$$

Combining these esimate with $\epsilon = \theta/4$, we get uniformly for all small h:

$$\frac{\theta}{4}\int \zeta^2|\delta_k^2\nabla u|^2\ \mathrm{d}x \leq C\int_U f^2 + u^2 + |\nabla u|^2\ \mathrm{d}x.$$

Thus by difference quotient theorem, it follows that $\nabla u \in H^1_{loc}$ and thus $u \in H^2_{loc}$ with $\|u\|_{H^2} \leq C(\|f\|_{L^2} + \|u\|_{H^1})$.

Final step 7: deduce that $\|u\|_{H^1(W)} \leq (\|f\|_{L^2} + \|u\|_{L^2})$ using weak formulation and uniform ellipticity. □

Theorem 1.108 (Higher-interior regularity)**.** *Let $U \subset \mathbb{R}^n$ be open, bounded. Assume $a^{ij}, b^i, c \in C^{m+1}(U)$ and $f \in H^m(U)$. Suppose $u \in H^1(U)$ is a weak solution of $Lu = f$ in U. then $u \in H^{m+2}(U)$ and for each $V \subset\subset U$, $\|u\|_{H^{m+1}(V)} \leq C(\|f\|_{L^2} + \|u\|_{L^2(U)}$. If even $a^{ij}, b^i, c \in C^\infty)U)$ and $f \in C^\infty(\overline{U}$, then $u \in C^\infty(U)$.*

1.12 Eigenvalues and Eigenfunctions

Let $U \subset \mathbb{R}^n$ be open and bounded. We consider the BVP:

$$Lw = \lambda w,\ w = 0\ \in \partial U$$

We say that λ is an eigenvalue of L if there exists a $w \neq 0$ that satisfies this BVP. Here we assume $Lu = -\sum_{i,j}\partial_j(a^{ij}(x)\partial_i u)$ where $a^{ij} \in C^\infty(\bar{U})$ and the coefficients are symmetric. Then the associated bilinear form is:

$$B[u,v] = \sum_{i,j}\int_U a^{ij}(x)\partial_i u(\partial_j v)\ \mathrm{d}x.$$

This satisfies $B[u,v] = B[v,u]$ for all $u, v \in H^1_0$ and L is formally symmetric.

Theorem 1.109 (Eigenvalues of symmetric elliptic operators).

- *Each eigenvalue of L is real.*
- *Repeating each eigenvalue according to its (finite) multiplicity, We have $\Sigma = \{\lambda_k\}_1^\infty$ which is non decreasing.*
- *There exists and orthonormal basis $\{w_k\}_1^\infty$ of $L^2(U)$ where $w_k \in H_0^1(U)$ is an eigenfunctions corresponding to λ_k.*

Proof. Sketch By ellipticity condition, for all $u \in H_0^1$, $B[u,u] = \sum_{i,j} \int a^{ij}\partial_i u \partial_j u \ \mathrm{d}x \geq \theta \int |\nabla u|^s \ \mathrm{d}x$. A Poincare inequality gives coercivity. As usual, $B[u,u]$ is also bounded. Lax-Milgram furnishes a unique solution: for all $f \in L^2$, there is a unique u such that $B[u,v] = (f,v)_{L^2}$ for all $v \in H_0^1$. Then set $u := S(f)$. We have that $S : L^2 \to H_0^1 \subset L^2$ which is linear, bounded, and compact from previous lectures.

Claim: S is self-adjoint, ie, $(Sf,g) = (f,Sg)$ for all $f,g \in L^2$. Indeed, let $(f,g) \in L^2$ and let $Lu = f$ and $Lv = g$ in the weak sense, so that $u = Sf$ and $v = Sg$. Now

$$(Sf,g)_{L^2} = (u,g)_{L^2} = B[v,u]$$

$$(f,Sg)_{L^2} = (f,v)_{L^2} = B[u,v]$$

Since B is symmetric, we have that S is self-adjoint. Now the a assertion follows from the Spectral theorem for compact self-adjoint operators applied to S. Observe that $B[Sw,v] = (w,v)_{L^2}$, whence $B[w,v] = (\frac{1}{\lambda}w,v)_{L^2}$. □

Remark 1.110. Thanks to regularity theory from last lectures, we actually get $w_k \in C^\infty(U)$ (and $w_k \in C^\infty(\bar{U})$ assuming ∂U is smooth).

Main additional tool: Spectral Theorem for self-adjoint compact operators.

Theorem 1.111. *Let H be a separable Hilbert space and let $S : H \to H$ be linear, compact, self-adjoint. Then there exists countable orthonormal basis of H consisting of eigenvectors of S. Moreover, for a compact operator all non-zero eigenvalues have finite multiplicity and and 0 is the only possible accumulation point.*

1.13 Linear parabolic PDEs

Let $U \subset \mathbb{R}^n$ be open and bounded and set $U_T := [0,T] \times U$ for $T > 0$. Goal: prove existence and uniqueness of weak-solutions and discuss regularity to the IVP/BVP.

$$\partial_t u + Lu = 0 \in U; \quad u = 0 \in \partial U; u = g, \quad t = 0.$$

where $(Lu)(t,x) := -\sum_{i,j=1}^n \partial_j(a^{ij}(t,x), \partial_i u) + \sum_1^n b^i(t,x)\partial_i u + c(t,x)u$. Generalization of $\partial_t u - \Delta u = f$.

1.13.1 Bochner Integral

Definition 1.112. Let $I \subset \mathbb{R}$ be some interval of finite length w.r.t Lebesgue measure on the line.

1. Let X be a Banach space with norm $\|\cdot\|$. We say that $f : I \to X$ if there exists some $n \in \mathbb{N}$, Lebesgue measurable sets E_j and $x_1, \ldots, x_n \in X$ such that:
$$f(t) = \sum_{n=1}^n \chi_{E_n}(t)x_n$$
2. The Bochner integral of a simple function is:
$$\int f(t) \ \mathrm{d}t := \sum_1^N |E_n|x_n.$$

Lemma 1.113.

- *The above definition is independent of the choice of E_n, ie, well-defined*
- *For all simple functions, $f : I \to X$, the function $t \mapsto \|f(t)\|$ is Lebesgue measurable.*
- $\|\int_I f(t) \ \mathrm{d}t\| \leq \int \|f(t)\| \ \mathrm{d}t$.

Definition 1.114. $f : I \to X$ is (strongly) measurable or Bochner measurable if there exists $\{f_n\}_1^\infty$, simple functions, such that for a.e. $t \in I$, $\|f(t) - f_n(t)\| \to 0$. In short, $f_j \to f$ a.e.

Lemma 1.115. *Let $f : I \to X$ be strongly measurable with some approximating sequence $f_j \to f$ a.e. The, for all $j \in \mathbb{N}$, $t \mapsto \|f_j(t) - f(t)\|$ is Lebesgue measurable.*

Proof. Note $\|f_j(t) - f(t)\| = \lim_{k\to\infty} \|f_j(t) - f_k(t)\|$ for a.e. t. □

Definition 1.116. $f : I \to X$ is Bochner integrable if f is strongly measurable and there exists $\{f_j\}_1^\infty$, simple functions approximating sequence of f (ie, $f_j \to f$) and $\lim_{j\to\infty} \int_I \|f_j(t) - f(t)\| \text{ d}t = 0$. In this case, we define:

$$\int_I f(t) \text{ d}t = \lim_{j\to\infty} \int_I f_j(t) \text{ d}t.$$

Lemma 1.117. *The Bochner integral is well-defined, that is, the limit exists and is independent of choice of approximating sequence.*

Proof. Existence of limit. We show that $\int_I f_j(t) \text{ d}t$ is Cauchy in X. Indeed:

$$\| \int_I f_j(t) \text{ d}t - \int_I f_k(t) \text{ d}t\| \leq \int_I \|f_j(t) - f_k(t)\| \text{ d}x \leq \int_I \|f_j(t) - f(t)\| \text{ d}t + \int_I \|f_k(t) - f(t)\| \text{ d}t \to 0.$$

Suppose $\{g_j\}_j$ is another approximating sequence. Then:

$$\| \int_I f_j(t) \text{ d}f - \int g_j(t) \text{ d}t\| \leq \int_I \|f_j(t) - f(t)\| \text{ d}t + \int_I \|g_j(t) - f(t)\| \text{ d}t \to 0.$$

□

Theorem 1.118 (Properties of Bochner integral)**.**

- *$\mathcal{L}(I, X) := \{f : I \to X : f$ if Bochner integrable $\}$ is a vector space*
- *$f \in \mathcal{L}(I, X)$ iff $f : I \to X$ is measurable and $t \mapsto \|f(t)\|$ is Lebesgue -integrable.*
- *For all $f \in \mathcal{L}(I, X)$:*

$$\| \int_I f(t) \text{ d}t\| \leq \int_I \|f(t)\| \text{ d}t$$

- *Let X, Y be Banach spaces, $T : X \to Y$ be a bounded linear operator and $f \in \mathcal{L}(I, X)$. Then Tf is a Bochner integrable function from I to Y and*

$$\int_I (Tf)(t) \text{ d}t = T(\int_T f(t) \text{ d}t).$$

Proof.

1. Yep.
2.
3. (=¿) If f is Bochner integrable, it is measurable by definition. The second condition that $t \mapsto \|f(t)\|$ is Lebesgue integrable follows form the fact that it is Lebesgue measurable and

$$0 \leq \|f(t)\| \leq \|f - f_j\| + \|f_j\|.$$

The definition of Bochner integrable therefore gives the result.

(¡=) Let $\{f_j\}_j$ be an approximating sequence of the measurable function f, ie, for a.e. $t \in I$ $\|f_j(t) - f(t)\| \to 0$. Fix $\epsilon > 0$ set

$$g_j(t) := \begin{cases} f_j(t) & \|f_j(t)\| \leq (1+\epsilon)\|f(t)\| \\ 0 & else \end{cases}$$

Since $t \to \|f_j(t)\|$ and $t \to \|f(t)\|$ are Lebesgue measurable

$$\{t \in I : \ \|f_j(t)\| \leq (1+\epsilon)\|f(t)\|\} := A_j$$

$t \mapsto g_j(t) := f_j(t)\chi_{A_j}(t)$ is a simple function with

- $\|g_j - f\| \to 0$
- $\|g_j - f\| \leq \|g_j\| + \|f\| \leq (2+\epsilon)\|f\|$ and so are Lebesgue integrable b.c. $t \mapsto \|f(t)\|$ is.
- $\lim_{j\to\infty} \int_I \|g_j(t) - f(t)\| \text{ d}t = 0$ by DCT. So f is Bochner integrable.

4. Using the g_j from before:

$$\| \int_I f(t) \text{ d}t\| = \| \lim_j \int_I g_j(t) \text{ d}\| = \lim_j \| \int_I g_j(t) \text{ d}t\| \leq \limsup \int_I \|g_j(t)\| \text{ d}t \leq (1+\epsilon) \int_I \|f(t)\| \text{ d}t.$$

$\epsilon > 0$ is arbitrary.

□

More practical characterization of strong measurability.

Definition 1.119. Let $f : I \to X$

- f has essentially separable image if there exists a Lebesgue-null set $N \subset I$ such that $\{f(t) : t \in I/N\}$ is separable.
- f is weakly measurable if for all $\xi \in X^*$, $I \to \mathbb{C}$, $t \mapsto \xi(f(t))$ is Lebesgue-measurable.

Theorem 1.120 (Petti's)**.** *Let $f : I \to X$. Then f is measurable iff f is weakly measurable and has an essentially separable image.*

Proof. (=¿) HW

(¡=) See section 4 of Chapter 5 in Yosida "Functional Analysis". □

Lemma 1.121. *For any norm space Y, Y is complete iff $\sum \|y_n\| < \infty$ implies $\lim \sum^N y_n := y$ exists.*

Definition 1.122. For $p \in [1, \infty)$ let

$$L^p(I, X) := \{f : I \to X : \text{ measurable and } \int_I \|f(t)\|^p \text{ d}t < \infty\}.$$

$$L^\infty(I, X) := \{f : I \to X : \text{ measurable } ess \sup_{t \in I} \|f(t)\| < \infty\}$$

Theorem 1.123. *$L^p(I, x)$ is a Banach space w.r.t the norm.*

Proof. Let $\{f_j\}_j \in L^p(I, X)$ with $\sum \|f_j\|_{L^p(I,X)} < \infty = \| \, \|f(t)\| \, \|_{L^p}$. Call $\phi_j(t) := \|f(t)\|$. So we have $\sum_1^\infty \|\phi_j\|_p < \infty$. But the lemma, there exists some $\phi \in L^p(I)$ such that $\phi = \sum_1^\infty \phi_j$ in L^p. Then there must exist a Lebesgue null set $N \subset I$ such that $\sum_j \phi_j(t) = \phi(t) < \infty$ for $t \in I/N$. Since X is complete, by Lemma, we have that $\sum_1^\infty \|f_j(t)\| = \phi(t) < \infty$ implies that for all $t \in I/N$ there exists some $f(t) \in X$ such that $\sum_j f_j(t) = f(t)$. Now we must show that $f \in L^p(I, X)$ and $\| \sum f_j - f\| \to 0$ in $L^p(I, X)$. Then the assertion will follow from the lemma.

Claim: f is measurable, we use Pettis' theorem.

- f as ess. sep. image since Pettis applied to each f_j gives that for all j there exists a Lebesgue null set N_j such that $f(I/N_j) := X_j$ is separable. And so $\{f(t) : t \in I/cupN_j\} \subset \overline{span(\cup_j X_j)}$ separable.
- For weakly measurable: for $\xi \in X^*$, for all $t \in I/N$

  $$\xi(f(t)) = \xi(\sum_1^\infty f_j(t)) = \lim \sum_1^N \xi(f_j(t))$$

  this is Lebesgue measurable since it is the limit of Lebesgue measurable functions.

Now $f \in L^p(I, X)$ follows from $\|f\|_{L^p(I,X)} \leq \|f - \sum_{j=1}^n f_j\|_{L^p(I,X)} + \|\sum_{j=1}^n f_j\|_{L^p(I,X)} < \infty$ since $\|f - \sum_1^n f_j\|_{L^p(I,X)} \leq \sum_{j=n+1}^\infty \|\phi_j\|_{L^p(I)} < \infty$. □

Theorem 1.124. *For $p \in [1, \infty)$, $C(I, X)$ is dense in $L^p(I, X)$.*

Proof. Step 1: Step functions are dense in $L^p(I, X)$. Let $f \in L^p(I, X)$. As in the earlier proof of the theorem, there exists $\{g_j\}_1^\infty$ of simple functions such that $\|g_j\| \leq 2\|f\|$ for all j and

$$\int_I \|g_j - f\| \text{ d}t.$$

So $\|g_j - f\|^p \to 0$ a.e. and $\|g_j - f\|^[p \leq 3^p \|f\|^p$. Thus by DCT $\int \|g_j - f\|^p \text{ d}t \to 0$. Thus simple functions are dense.

Step 2: Approximate simple function by those from $C(I, X)$. Consider

$$f = \sum_n^n \chi_{B_n} \phi_n$$

be simple with $\phi_n \neq 0$. Then For all $\epsilon > 0$ there exists $\gamma_n \in C(I)$ such that $\|\gamma_n - \chi_{B_n}\|_p \leq \frac{\epsilon}{N\|\phi_n\|}$. Then $\gamma_n \phi_n \in C(I, X)$ and $g = \sum_1^N \phi_n \gamma_n \in C(I, X)$ and

$$\|g - f\|_{L^p(I,X)} \leq \sum_{n=1}^N \|(\gamma_n - \chi_{B_n} \phi_n\|_{L^p(I,X)} < \epsilon.$$

Definition 1.125. Let $I \subset \mathbb{R}$ be an interval and

$$C^1(I,X) := \{f \in C(IX):\ \forall t \in I,\ \frac{f(t+h)-f(t)}{h} \text{ converges in X as } h \to 0 \text{ and } t \mapsto f'(t) \in C(I,X)\}.$$

Theorem 1.126. *1. If $f_n \to f$ in $L^p(I,X)$, then there exists a subsequence f_{n_k} such that $f_{n_k}(t) \to f(t)$ for a.e. $t \in I$*

2. *Fundamental theorem: for all $f \in C^1(I,X)$ and for all $s,t \in I$:*

$$f(t) = f(s) + \int_s^t f'(\tau)\ \mathrm{d}\tau.$$

3. *Let $\{f_n\}_n \subset \mathcal{L}(I,X)$ with $f_n(t) \to f(t)$ in X for a.e. $t \in I$ and suppose there exists a $g \in L^1(I)$ such that $\|f_n\| \leq g$ a.e. for all n then $f \in \mathcal{L}(I,X)$ and*

$$\int_I f_n(t)\ \mathrm{d}t \to \int_I f(t)\ \mathrm{d}t.$$

4. *Let $\eta in C_c^\infty(-\epsilon,\epsilon), f \in L^1(I,X)$. Then $\eta * f \in C^\infty(I_\epsilon, X)$ with $I_\epsilon := \{t \in I : dist(t,\partial I) \geq \epsilon\}$ and*

$$\frac{d^k}{(dt)^k}(\eta * f) = \frac{d^k\eta}{(dt)^k} * f.$$

5. *X is separable implies $L^p(I,X)$ is separable for $1 \leq p < \infty$.*

□

1.14 Weak Theory of linear parabolic equations

Consider the IVP/BVP for an unkown function $u : U_T \to \mathbb{R}$

$$\begin{cases} \partial_t u + Lu = f & \in U_T \\ u = 0 & \in [0,T] \times \partial U \\ u = g & @t = 0 \end{cases}$$

with

$$Lu := -\sum_{i,j} \partial_j(a^{ij}(t,x)\partial_i u) + \sum_{i=1}^n b^i(t,x)\partial_i u + c(t,x)u.$$

Definition 1.127. 1. $\partial_t + L$ is uniformly parabolic on U_T iff there exists some $\theta > 0$ such that for all $\xi \in \mathbb{R}^n$ and $(t,x) \in U_T$, $\sum_{i,j=1}^n a^{ij}(t,x)\xi_i\xi_j \geq \theta|\theta|^2$

2. The time dependent bilinear form associated with L is defined by

$$B[u,v;t] := \int_U \left(\sum_{i,j=1}^n a^{i,j}(t,x)\partial_i u \partial_j v + \sum_{i=1}^n b^i(t,x)\partial_i uv + c(t,x)uv \right)\ \mathrm{d}x.$$

for $u,v \in H_0^1(U)$ and a.e. $t \in [0,T]$.

Definition 1.128. Let $f \in L^2([0,T], L^2(U))$ and $g \in L^2(U)$. A function $u \in L^2([0,T], H_0^1)$ with weak derivative $u' \in L^2([0,T], H^{-1}(U))$ is a weak solution to the problem if the following two things hold:

1. $\langle u'(t), v\rangle + B[u(t), v(t), t] = (f(t), v)_{L^2}$ for all $v \in H_0^1(U)$ and a.e. $t \in [0,T]$.

2. $u(0) = g$.

Note 1.129. Here we are working with a Gelfand triple $(V, \mathcal{H}, V^*)$, weak solution $u \in \mathcal{H}^1$ and last time we saw that $\mathcal{H}^1 \subset C([0,T], H_0^1)$.

Heuristics: suppose u, f are smooth. Multiply by $\partial_t u + Lu = f$ by v and integrate.

Our goal is to construct a weak solution to the IVP/BVP. The strategy is to use Galerkin's method:

Step 1: Finite dimensional approximation; get ODEs with solutions $\{u_m\}_m$

Step 2: Energy estimates: $\{u_m\}_m \subset \mathcal{H}^1$ uniformly bounded

Step 3: Extract a weakly convergent subsequence and show that the weak limit is a weak solution.

Step 4: Uniqueness of weak solutions

First, Step 1: In the following, $\{w_k\}_1^\infty$ be a sequence of smooth functions such that it is an orthonormal basis of L^2 and an orthogonal basis of H_0^1. To this end, invoke our theorem about eigenvalues of symmetric elliptic operators applied to $L = -\Delta$. Fix $m \in \mathbb{N}$. The idea is to construct a solution u_m to the "projection" of the problem to the first m elements of the basis. We look for a function $u_m : [0,T] \to H_0^1$ of the form $u_m(t) = \sum_{k=1}^m d_m^k(t) w_k$ where the coefficients $d_m^k : [0,T] \to \mathbb{R}$ satisfies $d_m^k(0) = (g, w_k)_{L^2}$ and $(u_m', w_k)_{L^2} + B[u_m(t), w_k, t] = (f(t), w_k)_{L^2}$ for a.e. t and $k = 1, \ldots, m$.

Theorem 1.130. *For all $m \in \mathbb{N}$ there exists a unique $u_m(t)$ of the form $\sum_{k=1}^m d_m^k(t) w_k$ satisfies these properties.*

Proof. If $u_m(t)$ is of the form $u_m(t) := \sum_{l=1}^m d_m^l(t) w_l$, then

$$(u_m'(t), w_k) = \sum_{l=1}^m d_m^l(t)(w_l, w_k) = (d_m^k)'(t)$$

and

$$B[u_m(t), w_k, t] = \sum_{l=1}^m \underbrace{B[w_l, w_k; t]}_{e^{lk}(t)} d_m^l(t).$$

We also right:

$$f^k(t) := (f(t), w_k)$$

Then the weak form becomes the linear system of ODEs:

$$(d_k^l)'(t) + \sum_{l=1}^m e^{kl}(t) d_m^l(t) = f^k(t)$$
$$d_m^k(0) = (g, w_k)_L.$$

By Caratheodory existence theory for such ODEs, there exists a unique absolutely continuous solution $d_m(t) = (d_m'(t), \ldots, d_m^m(t))$ satisfying the above requirements for a.e. t. Then, $u_m(t) := \sum_{k=1}^m d_m^k(t) w_k$ has the desired properties. For more details, see Coddington-Levinson: "Theory of ODEs" 1955, [Chapter 2, Theorem 1.1]. □

Step 2: Energy estimate.

Theorem 1.131. *There exists $C \equiv C(U, T, L)$ such that for all $m \in \mathbb{N}$:*

$$\max_{0 \le t \le T} \|u_m(t)\|_{L^2(U)} + \|u_m\|_{L^2([0,T], H_0^1)} + \|u_m'\|$$

Recall:

Step 1: Galerkin Method

Finite dimentional approximation:

$$u_m : [0,T] \to H_0^1$$

with

$$u_m(t) := \sum_{k=1}^m d_m^k(t) w_k$$

$$d_m^k(0) = (g, w_k)$$

$$(u_m'(t), w_k)_{L^2} + B[u_m(t), w_k, t] = (f(t), w_k).$$

Theorem 1.132. *For all $m \in \mathbb{N}$, there exists a unique $u_m(t)$ of the above form satisfying these properties.*

New:

Theorem 1.133. *There exists $C \equiv C(U, T, L)$ such that for all $m \in \mathbb{N}$*

$$\max_t \|u_m(t)\|_{L^2(U)} + \|u_m\|_{L^2([0,T], H_0^1(U))} + \|u_m'\|_{L^2([0,T], H_0^1)} \le C \left(\|f\|_{L^2([0,T], L^2(U))} + \|g\|_{L^2} \right)$$

Proof. Multiply variational form by $d_m^k(t)$ and sum $k = 1, \ldots, m$. Recall that $u_m = \sum_{k=1}^m d_M^k(t) w_k$ to get:

$$(u_m'(t), u_m(t))_{L^2} + B[u_m(t), u_m(t); t] = (f, u_m(t))_{L^2}.$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u_m(t)\|_{L^2}^2) + B[u_m(t), u_m(t), t] \le \frac{1}{2}\|f\|_{L^2(U)}^2 + \frac{1}{2}\|u_m(t)\|_{L^2}^2.$$

Using that L is uniformly elliptic, there exists $\beta > 0, \gamma > 0$ so that $B[u_m(t), u_m(t), t] \ge \beta\|u_m(t)\|_{H_0^1}^2 - \gamma\|u_m(t)\|_{L^2}^2$. Thus, we get:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u_m(t)\|_{L^2}^2) + 2\beta\|u_m\|_{H_0^1}^2 \le C(\|u_m\|_{L^2}^2 + \|f\|_{L^2}^2).$$

We can use Gronwall: $\eta(t) := \|u_m(t)\|_{L^2}^2,\ \xi(t) := \|f(t)\|_{L^2}^2$. Then $\eta'(t) \le C(\eta(t) + \xi(t)$. Gronwall implies that $\eta(t) \le e^{\int_0^t C\ \mathrm{d}s}\left(\eta(0) + \int_0^t \xi(s)\ \mathrm{d}s\right) \le e^{Ct}\left(\eta(0) + \int_0^t \xi(s)\ \mathrm{d}s\right)$

Use $\|u_m(0)\|_{L^2}^2 = \|\sum_1^m (w_k, u(0))w_k\|_{L^2}^2 = \sum_{k=1}^m |(w_k g)|^2 \le \|g\|_{L^2}$ to get:
$\le C\left(\|g\|_{L^2} + \|f\|_{L^2([0,T],L^2)}\right)$. Thus we have that $\max_t \|u_m(t)\|_{L^2(U)} \le C\left(\|f\|_{L^2([0,T],L^2)} + \|g\|_{L^2}\right)$.

To get the second term, we integrate the ODE in time:

$$\|u_m(T)\|_{L^2}^2 - \|u_m(0)\|_{L^2}^2 + 2\beta\int_0^T \|u_m(t)\|_{H_0^1}^2\ \mathrm{d}t \le C\int_0^T \left(\|u_m(t)\|_{L^2}^2 + \|f(t)\|_{L^2}^2\right)\ \mathrm{d}t$$

We bound the right hand side using the maximum inequality we just derived and the fact that $\|u_m(0)\|^2 \le \|g\|^2$, we conclude:

$$\|u_m\|_{L^2([0,T],H_0^1)}^2 \le C\left(\|g\|_{L^2}^2 + \|f\|_{L^2([0,T],L^2)}^2\right).$$

Finally, to bound $\|u_m'(t)\|_{L^2([0,T],H^{-1})}$: Fix any $v \in H_0^1$ with $\|v\|_{H_0^1} \le 1$. We write: $v = v^1 + v^2$ where $v_1 \in$ *span*$\{w_1, \ldots, w_m\}$ where $(v^2, w_k) = 0$ for $k = 1, \ldots, m$ and $v^1 \le \|v\|_{H_0^1} \le 1$. From the variational equation, at each m:

$$(u_m'(t), w_k) + B[u_m(t), w_k, t] = (f(t), w_k)$$

$$(u_m', v^1) + B[u_m, v^1, t] = (f(t), v^1)$$
$$(u_m'(t), v^1) = \langle u'm(t), v^1\rangle = -B[u_m, v^1, t] + (f(t), v^1).$$

Thus:

$$|\langle u_m'(t), v^1\rangle| \le |B[u_m, v^1, t]| + |(f(t), v^1)| \le C\max_t \|u_m(t)\|_{H_0^1}\|v^1\|_{H_0^1} + C\|f(t)\|_{L^2} + C\|v^1\|_{H_0^1}.$$

Then using $\|v^1\| \le 1$:

$$|\langle u_m'(t), v^1\rangle| \le |B[u_m, v^1, t]| + |(f(t), v^1)| \le C\max_t \|u_m(t)\|_{H_0^1} + C\|f(t)\|_{L^2} + C$$

For any $v \in H_0^1$:

$$\langle u_m'(t), v\rangle = (u_m'(t), v)_{L^2} \implies \|u_m'(t)\|_{H^{-1}} = \sup_{\|v\|_{H_0^1} \le 1} |\langle u_m'(t), v\rangle| \le C(\|u_m(t)\|_{H_0^1} + \|f(t)\|_{L^2}).$$

Now square and integrate from 0 to T:

$$\int_0^T \|u_m(t)\|_{H^{-1}}^2\ \mathrm{d}t \le C\left(\int_0^T \|u_m(t)\|_{H_0^1}^2\ \mathrm{d}t + \int_0^T \|f(t)\|_{L^2}^2\ \mathrm{d}t\right)$$

Using the L^2 bound once again, we get:

$$\int_0^T \|u_m(t)\|_{H^{-1}}^2\ \mathrm{d}t \le C\left(\|g\|_{L^2}^2 + \|f\|_{L^2([0,T],L^2)}\right).$$

This completes the proof. □

For step 3: Existence via extracting weakly convergent subsequence.

Theorem 1.134. *There exists a weak solution to the IVP/BVP.*

Proof. □

Lemma 1.135. *Let V be a reflexive Banach space. Then $L^p([0,T];V)$ is reflexive for $1 < p < \infty$ with*

$$(L^p([0,T];V) \sim L^q([0,T],V^*))$$

Step 3: Existence.

By step 2, $\{u_m\}_m \subset L^2([0,T],H_0^1)$ is uniformly bounded and $\{u_m'\}_m \subset L^2([0,T],H^{-1})$ is uniformly bounded. By weak compactness, there exists a weak subsequence $\{u_{m_l}\} \subset \{u_m\}$ and a function $u \in L^2([0,T],H_0^1)$ with $u' \in L^2([0,T],H^{-1})$ such that $u_{m_l} \to u$ weakly in $L^2([0,T],H_0^1)$ and $u_{m_l}' \to u'$ in $L^2([0,T],H^{-1})$. The fact that the weak derivatives coincide follows from the definition of a weak derivative. By slight abuse of notation, in what follows we write $m = m_l$. For every $m \in \mathbb{N}$, we have $(u'm(t), w_k) + B[u_m(t), w_k, t] = (f(t), w_k)$ for a,e. t and all $k = 1, \ldots, m$.

For every $n \in \mathbb{N}$, define by $V_n := span\{w_1, \ldots, w_n\}$. Thus, for any test function $\phi \in C^1([0,T])$ with $\phi(T) = 0$, for all $n \in \mathbb{N}$ for all $v \in V_n$ and $m \geq n$:

$$\int_0^T \langle u'm(t), v\rangle \phi(t)\ \mathrm{d}t + \int_0^T B[u_m(t), v, t]\phi(t)\ \mathrm{d}t = \int_0^T (f(t), v)\phi(t)\ \mathrm{d}t.$$

As $m \to \infty$ we have:

$$\int_0^T \langle u'(t), v\rangle \phi(t)\ \mathrm{d}t.$$

Claim: For any fixed $w \in L^2([0,T],H_0^1)$, $\int_0^T B[\cdot, w(t), t]\ \mathrm{d}t$ is weakly continuous in $L^2([0,T],H_0^1)$. By the claim, the identity holds for all $v \in \cup_1^\infty V_n$. By continuity and dominated convergence convergence, this holds for all $v \in \overline{\cup_1^\infty V_h} = H_0^1$. This gives existence by fundamental lemma of variational calculus. for a.e. t and all $v \in H_0^1$, $\langle u'(t), v\rangle + B[u(t), v, t] = (f(t), v)$. It remains to show $u(0) = g$. From our construction, $u_m(0) = P_m(g)$, where P_m is the projection to the finite dimensional space.

On the one hand, we can integrate by parts in time the full case (not finite dimensional). Note that

$$\int_0^T \langle u'(t), \phi(t)v\rangle\ \mathrm{d}t = \int_0^T (u(t), \phi'(t)v)\ \mathrm{d}t + \underbrace{(u(t), \phi(t)v)|_0^T}_{=-(u(0),\phi(0)v}.$$

Thus, from the identity:

$$\int_0^T (-u(t), v)\phi'(t)\ \mathrm{d}t + B[u(t), v, t]\phi(t)\ \mathrm{d}t = \int_0^T (f(t), v)\phi(t) + (u(0), v)\phi(0).$$

On the other hand, we can integrate by parts in time in the finite dimensional case, we get. Note for all $v \in V_h$, $m \geq n$:

$$\int_0^T (-(u_m(t), v)\phi'(t) + B[u_m(t), v, t]\phi(t)\ \mathrm{d}t = \int_0^T (f(t), v)\phi(t)\ \mathrm{d}t + (u_m(0), v)\phi(0).$$

The last term in the limit is $(g, v)\phi(0)$. So we take the limit to infinity and subtracting the two cases, we get that $(u(0), v)\phi(0) = (g, v)\phi(0)$. This means that $u(0) = g$.

Now to prove the claim:

Let $\{u_m \to 0\}$ weakly in $L^2([0,T],H_0^1)$, ie, for all $l \in L^2([0,T],H^{-1})$,

$$\int_0^T [l(t)](u_m(t))\ \mathrm{d}t \to 0.$$

Need to show:

$$\int_0^T B[u_m, w(t), t]\ \mathrm{d}t \to 0$$

Only discuss the top-order term in the bilinear form:

$$\int_0^T \int_U a^{ij}(t,x)\partial_i u_m(t,x)\partial_j w(t,x)\ \mathrm{d}x\ \mathrm{d}t.$$

Define the middle part as l_w. We want to show $l_w \in L^2([0,T],H^{-1}(U))$. But this follows by crudely taking absolute values, applying Holder, and getting $\|l_w(t)\|_{H^{-1}} \leq C\|\nabla w(t)\|_{L^2}$. To see that l_w is actually in L^2, we just take the integral, use the bound, and observe that $w \in L^2([0,T],H_0^1)$. This proves the claim.

Step 4: Uniqueness:

Theorem 1.136. *There exists at most one weak solution to the IVP/BVP.*

Proof. By linearity, it suffices to check that the only weak solution to our problem with 0 data is the 0 solution. A weak solution u to the IVP/BVP with 0 data satisfies:

$$\langle u'(t), v\rangle + B[u(t), v, t] = 0.$$

Now insert $u(t) = v$. Then, since L is uniformly parabolic, there exists constants $\beta, \gamma > 0$ such that $B[u(t), u(t), t] \geq \beta\|u(t)\|_{H_0^1}^2 - \gamma\|u(t)\|_{L^2}^2$

$$\langle u'(t), u(t)\rangle + B[u(t), u(t), t] = 0 \implies \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|u(t)\|_{L^2} \leq C\|u(t)\|_{l^2}^2.$$

By Gronwall, this implies that $\|u(t)\|_{L^2}^2 \leq e^{Ct}\|u(0)\|_{L^2}^2 = 0$. Hence for $0 \leq t \leq T$ and $u(t) \equiv 0$. $\square$

Regularity:

Heuristic computation: Assume $u(t,x)$ is a smooth and sufficiently decaying solution to $\partial_t u - \Delta u = f$ in $[0,T] \times \mathbb{R}^n$ with $u(0) = g$. Then we have for $0 \leq t \leq T$:

$$\int_{\mathbb{R}^n} f^2 \, \mathrm{d}x = \int_{\mathbb{R}^n} (\partial_t u - \Delta u)^2 \, \mathrm{d}x = \int_{\mathbb{R}^n} (\partial_t u)^2 - \underbrace{2\int \partial_t u \Delta u \, \mathrm{d}x}_{-- \frac{1}{2}\partial_t \int_{\mathbb{R}^n} |\nabla u| \, \mathrm{d}x} + \int (\Delta u^2) \, \mathrm{d}x$$

Thus

$$= \int_{\mathbb{R}^n} (\partial_t u)^2 \, \mathrm{d}x + \partial_t \left(\int_{\mathbb{R}^n} |\nabla u|^2 \, \mathrm{d}x\right) + \|\nabla^2 u\|_{L^2}^2.$$

Now integrate over time:

$$\sup_t \int |\nabla u(t)|^2 \, \mathrm{d}x + \int_0^T \int_{\mathbb{R}^n} (\partial_t u)^2 + |\nabla^2 u|^2) \, \mathrm{d}x \, \mathrm{d}t \leq C\left(\int_{\mathbb{R}^n} |\nabla u(0)|^2 \, \mathrm{d}x + \int_0^T \int_{\mathbb{R}^n} f(t,x)^2 \, \mathrm{d}x \, \mathrm{d}t\right).$$

This sort of idea says that if $g \in H^1$, then we get that $u \in L^2([0,T], H_0^2)$ to get regularity.

Theorem 1.137.

- *Assume $g \in H_0^1$ and $f \int L^2([0,T], L^2(U))$.*
- *Suppose $u \in L^2([0,T], H_0^1)$, $u' \in L^2([0,T], H^{-1})$ is the weak solution to*

$$\partial_t u - Lu = f \quad u = 0 \quad u(0) = g$$

Then in fact $u \in L^2([0,T], H^2) \cap L^\infty([0,T], H_0^1)$ and there exist $C \equiv C(U,T,L)$ such that

$$ess \sup_t \|u(t)\|_{H_0^1} + \|u\|_{L^2([0,T],H^2} + \|u'\|_{L^2([0,T],L^2)} \leq C(\|g\|_{H^1} + \|f\|_{L^2([0,T],L^2)}.$$

1.15 Semi-group theorem

View a linear PDE

$$\partial_t w - Aw = 0, \quad u(0) = g$$

as a first order ODE for for a suitable (real) X-valued solution $w : [0,\infty) \to X$ where $A : dom(A) \to X$ is a linear operator (usually a partial differential operator, which is unbounded).

Suppose we have a unique solution $w : [0,\infty) \to X$ for any initial condition $g \in X$. Then write $w(t) = s(t)g$. This point of view suggest properties of the family of operators $\{S(t)\}_{t\geq 0}$:

- $w(0) = g \implies S(0) = Id$
- By uniqueness, we have the following relations for any $s, t \geq 0$:

$$S(t+s)g = S(t)S(s)g = S(s)S(t).$$

- Reasonable assumption: for all $g \in X$, the mapping $[0,\infty) \to X$ defined by $t \mapsto S(t)g$ is continuous.

The idea is to do this idea backward: what do we need to know about A to get a good family with such nice properties, for then we can talk about $w(t) := S(t)g$ as a solution.

Let X be a real Banach Space.

Definition 1.138.

- $\{S(t)\}_{t\geq 0} \subset BL(X)$ (bounded linear operators) is called a linear operator *semi-group* on X iff $S(0) = Id$ and $S(t+s) = S(t)S(s) = S(s)S(t)$

- $t \mapsto S(t)u$ is continuous for all $u \in X$, and a *contraction semi-group* if in addition $\|S(t)\|_{X \to X} \leq 1$ for all $t \in [0, \infty)$.
- Let $\{S(t)\}$ be a contraction semi-group on X. Define $D(A) := \{u \in X : \lim_{t\emptyset 0} \frac{S(t)u-u}{t}\}$ exists in X. For $u \in D(A)$, we set $Au := \lim_{t \to 0} \frac{S(t)u-u}{t}$. We call $A : D(A) \to X$ the (infinitesimal) generator of $\{S(t)\}$ and $D(A)$ is the domain of A. (To motivate this notion, think of a power series expansion of $S(t)$ and we pick up something that we want to call A, ie, the best linear term that approximates $S(t)$.

Throughout, $\{S(t)\}_{t \geq 0}$ will always denote a contraction semi-group.

Theorem 1.139 (Differential properties of semigroups)**.** • *Let $u \in D(A)$. Then, $S(t)u \in D(A)$ for all $t \geq 0$.*

- *$AS(t)u = S(t)Au$ for all $t \geq 0$*
- *$t \mapsto S(t)u$ is differentiable for all t.*
- $$\frac{\mathrm{d}}{\mathrm{d}t}(S(t)u) = AS(t)u \quad \forall t > 0$$
  .

Corollary 1.140. $S(t)u \in C^1([0, \infty), X)$ for all $x \in D(A)$.

Proof. • (1) and (2). Let $u \in D(A)$. Then, $\lim_{s \to 0} \frac{S(s)S(t)u - S(t)u}{s} (= AS(t)u)$. So:

$$AS(t)u := \lim_{s \to 0} \frac{S(s)S(t)u - S(t)u}{s} = \lim_{s \to 0} \frac{S(t)S(s)u - S(t)u}{s} \lim_{s \to 0} S(t) \left(\frac{S(s)u - u}{s} \right) = S(t) \underbrace{(\lim_{s \to 0} \frac{S(s)u - u}{s})}_{:= Au} = S(t)Au$$

This gives us (2) and one follows from the fact that the limit exists and we just apply $S(t)$.

- (3) and (4)

$$\lim_{s \to 0} \left(\frac{S(t+s)u - S(t)u}{s} - S(t)Au) \right) = \lim_{s \to 0} \left(\frac{S(t)S(s)u - S(t)u}{s} = S(t)Au \right) = S(t) \left(\lim_{s \to 0} \frac{S(s)u - u}{s} - Au \right) = 0.$$

Moreover, for $t > 0$, for small enough s

$$\lim_{s \to 0} \left(\frac{S(t-s)u - S(t)u}{-s} - S(t)Au \right) = \lim_{s \to 0} \left(S(t-s) \frac{S(s)u - u}{s} - S(t)Au \right)$$

Use $\|S(t-s)\| \leq 1$ for all $s > 0$ small.

□

Corollary 1.141. For a contraction semi group $\{S(t)\}_t$ with infinitesimal generator A, the function $w(t) := S(t)u$ is a differential solution to the IVP $\partial_t w - Aw = 0$ $w(0) = u$.

Theorem 1.142 (Properties of generators)**.** • *The domain $D(A)$ is actually dense in X*

- *The generator A is a closed operator.*

Proof. • Fix $u \in X$. Define $u^t := \int_0^T S(s)u \, \mathrm{d}s$ for $t > 0$ (Bochner integral). Claim: $\lim_{t \to 0} \frac{1}{t} u^t = u$ in X and $u^t \in D(A)$ for all $t > 0$. This would give the first point. For the first part of the claim: since $s \mapsto S(s)u$ is continuous and $S(0)u = u$, given $\epsilon > 0$, there exists a $\delta > 0$ such that $\|S(s)u - u\|_X < \epsilon$ when $t \in (0, \delta)$. Thus,

$$\|\frac{1}{t}u^t - u\| = \|\frac{1}{t} \int_0^t (S(s)u - u) \, \mathrm{d}s\| \leq \frac{1}{t} \int_0^t \|S(s)u - u\| \, \mathrm{d}s \leq \epsilon.$$

For the second claim: For any $r > 0$

$$\frac{S(r)u^t - u^t}{r} = \frac{1}{r}(S(r) \int_0^t S(s)u \, \mathrm{d}s - \int_0^t S(s)u \, \mathrm{d}s) = \frac{1}{r}(\int_0^t \underbrace{S(r)S(s)}_{= S(r+s)} u \, \mathrm{d}s - \int_0^t S(s)u \, \mathrm{d}s = \frac{1}{r}(\int_r^{t+r} S(\tilde{s})u \, \mathrm{d}\tilde{s} - \int_0^t S(s)u \, \mathrm{d}s)$$

Now take limit:

$$\lim_{r \to 0} \frac{S(r)u^t - u^t}{r} = \lim_{r \to 0} (\frac{1}{r} \int_t^{t+r} S(s)u \, \mathrm{d}s - \frac{1}{r} \int_0^r S(s)u \, \mathrm{d}s) = S(t)u - S(0)u = S(t)u - u.$$

Thus, $u^t \in D(A)$ since the limit exists with $Au^t = S(t)u - u$.

- Let $\{u_k\}_k k \subset D(A)$ with $u_k \to u$, $Au_k \to v$. By the above theorem, we have:
$$S(t)uu_k - u_k = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}(S(s)u_k)\ \mathrm{d}s = \int_0^t AS(s)u_k\ \mathrm{d}s = \int_0^t S(s)Au_k\ \mathrm{d}s.$$
Now let $k \to \infty$ and we use the assumptions to get (also using uniform bound on $S(t)$:
$$S(t)u - u = \int_0^t S(s)v\ \mathrm{d}s.$$
Thus, $\lim_{t\to 0} \frac{S(t)u-u}{t} = \lim_{t\to 0} \frac{1}{t}\int_0^t S(s)v\ \mathrm{d}s = S(0)v = v$. Thus, $u \in D(A)$ with $Au = v$.

Important to note here: we cannot apply the closed graph theorem since A is not an operator on the whole space, only $D(A)$. □

1.15.1 Resolvents

Definition 1.143. Let $A : D(A) \to X$ be a closed linear operator with domain $D(A) \subset X$.

1. A real number $\lambda \in \mathbb{R}$ belongs to the *resolvent* set, $\rho(A)$, of A if the operator $\lambda ID - A : D(A) \to X$ is invertible (one to one and onto)
2. If $\lambda \in \rho(A)$, $\mathbb{R}_\lambda : X \to D(A)$ is defined by $R_\lambda u = (\lambda Id - A)^{-1}u$.

Exercise: by closed graph theorem, $\mathbb{R}_\lambda$ is a bounded linear operator. Moreover, we have that $AR_\lambda u = R_\lambda Au$ if $u \in D(A)$.

Theorem 1.144.
- *If $\lambda, \mu \in \rho(A)$,*
$$R_\lambda - R_\mu = (\mu - \lambda)R_\lambda R_\mu.$$
which we call the resolvent identity and $R_\lambda R_\mu = R_\mu R_\lambda$.
- *If $\lambda > 0$, then $\lambda \in \rho(A)$ and $R_\lambda u = \int_0^\infty e^{-\lambda t}S(t)u\ \mathrm{d}t$. Then, $\|R_\lambda\| \le \frac{1}{\lambda}$.*

Proof.
- Exercise (look up functional textbook)
- Assuming the identity holds, we get
$$\|R_\lambda u\| \le \int_0^\infty \|e^{-\lambda t}S(t)u\|\ \mathrm{d}t \le \frac{1}{\lambda}\|u\|.$$
Now to prove the identity. We make the following claims, which we do not prove but are notes online:
$$(1): \quad I(u) := \int_0^\infty e^{-\lambda t}S(t)u\ \mathrm{d}t$$
is a well defined integral.
And secondly:
$$AI(u) = I(Au).$$
For any $\lambda > 0$, define the linear operator $\tilde{R}_\lambda : X \to X$ by
$$\tilde{R})_\lambda u := \int_0^\infty e^{-\lambda t}S(t)u\ \mathrm{d}t.$$
Fix $\lambda > 0$. For positive $h > 0$ and $u \in X$:
$$\frac{S(h)\tilde{R}_\lambda u - \tilde{R}_\lambda u}{h} = \frac{1}{h}\int_0^\infty e^{-\lambda t}(S(t+h)u - S(t)u\ \mathrm{d}t) = \frac{1}{h}\int_h^\infty e^{-\lambda(\tilde{t}-h}S(\tilde{t})u\ \mathrm{d}\tilde{t} - \frac{1}{h}\int_0^\infty e^{-\lambda t}S(t)u\ \mathrm{d}t.$$
$$= -e^{\lambda h}\frac{1}{h}\int_0^h e^{-\lambda\tilde{t}}S(\tilde{t})\ \mathrm{d}\tilde{t} + \frac{e^{\lambda h-1}}{h}\underbrace{\int_0^\infty e^{-\lambda\tilde{t}}S(\tilde{t})u\ \mathrm{d}\tilde{t}}_{\tilde{R}_\lambda u}.$$
Now:
$$\lim_{h\to 0}\frac{S(h)\tilde{R}_\lambda u - \tilde{R}_\lambda u}{h} = -u + \lambda\tilde{R}_\lambda u.$$
Thus $\tilde{R}_\lambda u \in D(A)$ and $A\tilde{R}_\lambda u = -u + \lambda\tilde{R}_\lambda u$.

Hence
$$(\lambda ID - A)\tilde{R}_\lambda u = u.$$
On the other hand, for an element $u \in D(A)$, $A\tilde{R})\lambda u = A\int_0^\infty e^{-\lambda t}S(t)u\ \mathrm{d}t = \int_0^\infty e^{-\lambda t}S(t)Au\ \mathrm{d}t = \tilde{R}_\lambda Au$. Thus, for all $u \in D(A)$, $\tilde{R}_\lambda(\lambda Id - A)u = (\lambda Id - A)\tilde{R}_\lambda u = u$. Hence, from these properties, we have that $\lambda I - A$ is invertible with inverse $\tilde{R})_\lambda$. □

Theorem 1.145 (Hille-Yosida)**.** *Let A be a closed, densely defined linear operator on a Banach space X. Then A is the contractor of a contraction semi-group iff $(0, \infty) \subset \rho(A)$ and $\|R_\lambda\| \leq \frac{1}{\lambda}$ for all $\lambda > 0$.*

Proof. One direction is just the last theorem.

Now we show that A is a generator. For $\lambda > 0$, define $A_\lambda := 0\lambda Id + \lambda^2 R_\lambda = -\lambda(\lambda I - A)R_\lambda + \lambda^2 R_\lambda = +\lambda A R_\lambda$. Thus, A_λ id bounded. Moreover, for all $u \in D(A)$:

$$\lim_{\lambda \to \infty} A_\lambda u = Au$$

because $\lambda R_\lambda u - u = AR_\lambda u = R_\lambda Au$. So

$$\|\lambda R_\lambda u - u\| \leq \|R_\lambda\| \|Au\| \leq \frac{1}{\lambda} \|Au\| \to 0.$$

Thus, $\lambda R_\lambda u \to u$ for all $u \in D(A)$. Since the domain of A is dense in X and since $\|\lambda R_\lambda\| \leq 1$ we get $\lambda R_\lambda u \to u$ for all $u \in X$. Thus, for $u \in D(A)$, $A_\lambda u = \lambda A R_\lambda u = \lambda R_\lambda (Au) \to Au$. □

Now define $S_\lambda(t)$, the approximating contraction semigroups. For any $\lambda > 0$, $S_\lambda(t) := e^{tA_\lambda} = \sum_{k=0}^{\infty} (tA_\lambda)^k/(k!)$ (which converges in operator because A_λ is bounded, and so converges in operator norm). Note $S_\lambda(t) = e^{tA_\lambda} = e^{t(-\lambda Id + \lambda^2 dR_\lambda} = e^{-\lambda t Id} e^{t\lambda^2 R_\lambda} = e^{-\lambda t} Id \sum_{k=0}^{\infty} \frac{(t\lambda^2 R_\lambda)^k}{k!}$. Thus:

$$\|S\lambda(t)\|_{op} \leq e^{-\lambda t} \sum_{k=0}^{\infty} \| \frac{(t\lambda^2 R_\lambda)^k}{(k!)} \|.$$

Since $\|t\lambda^2 R_\lambda\| \leq (t\lambda)^t$, we obtain:

$$\|S)\lambda(t)\| \leq e^{-\lambda t} e^{\lambda t}.$$

Moreover, from the definition $S_\lambda(t) := e^{tA_\lambda}$ we get for all $\lambda > 0$, $S_\lambda(t+s) = S_\lambda(t) S_\lambda(s)$, $S_\lambda(0) = Id$, and $t \mapsto S_\lambda(t)u$ is continuous for all $u \in X$. Thus, $\{S_\lambda(t)\}_{t \geq 0}$ for all $\lambda > 0$ is a contraction semigroup.

Now define $S(t)$ as limit of $S_\lambda(t)$ as $\lambda \to \infty$ (we show how to do this). Let $\lambda, \mu > 0$. Then $R_\lambda R_\mu = R_\mu R_\lambda$. Thus, $A_\lambda A_\mu = A_\mu A_\lambda$. This implies that $A_\mu S_\lambda(t) = S_\lambda(t) A_\mu$. We show that for any $u \in X$, and any $t \in [0, \infty)$, $\{S_\lambda(t)u\}_{\lambda > 0}$ is a Cauchy sequence as $\lambda \to \infty$.

For $u \in D(A)$, we consider $S_\lambda(t)u - S_\mu(t)u = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}(S_\mu(t-s)S_\lambda(s)u)\ \mathrm{d}s$. Compute the integrand a bit further: $(-1)A_\mu S_\mu(t-s)S_\lambda(s)u + S_\mu(t-s)S_\lambda(s)u$. So we get:

$$= \int_0^t (S_\mu(t-s)S_\lambda(s)(A_\lambda - A_\mu)u)\ \mathrm{d}s.$$

Hence

$$\|S_\lambda(t)u - S_\mu(t)u\| \leq \int_0^t \|S_\mu(t-s)S_\lambda(s)(A_\lambda - A_\mu)s\|\ \mathrm{d}s \leq \int_0^t \|(A_\lambda - A_\mu)u\|\ \mathrm{d}s = t\|(A_\lambda - A_\mu)u\|$$

Since $A_\lambda u \to 0$ as $\lambda \to \infty$ for any $u \in X$, the rhs is a Cauchy sequence in λ. Note that this convergence is uniform in t on compact time intervals. Thus, for $u \in D(A)$ and $t > 0$, we can define $S(t)u := \lim_{\lambda \to \infty} S_\lambda(t)u$. This was for $u \in D(A)$, so density gives us a definition for $u \in X$.

The family of operators $\{S(t)\}_{t \geq 0}$ is also a contraction semigroup: These properties transfer from those for our approximating $\{S_\lambda(t)\}_{t \geq 0}$ using in particular that the pointwise limit is uniform in t on compact time intervals.

The final step is to show that A is the generator of our derived $\{S(t)\}_{t \geq 0}$. Let B be the generator of the contraction semigroup. We show $B = A$. For all $t \geq 0$, $\lambda > 0$, $u \in D(A)$:

$$S_\lambda(t)u - u = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}(S_\lambda(s)u)\ \mathrm{d}s = \int_0^t S_\lambda(s)A_\lambda u\ \mathrm{d}s.$$

Moreoever, we have that $\|S_\lambda(s)A_\lambda u - S(s)Au\| \leq \|S_\lambda(s)\| \|A_\lambda u - Au\| + \|(S_\lambda(s) - S(s))Au\| \to 0$ as $\lambda \to \infty$ uniformly in s on compact time intervals. Thus, taking $\lambda \to \infty$ in the lhs, we achieve: $S(t)u - u = \int_0^t S(s)Au\ \mathrm{d}s$. Hence, for $t > 0$, $u \in D(A)$:

$$\frac{1}{t}(S(t)u - u) = \frac{1}{t} \int_0^t S(s)Au\ \mathrm{d}s.$$

As $t \to 0$, we get:

$$Bu = S(0)Au = Au.$$

Hence, $D(A) \subset D(B)$ and $B|_{D(A)} = A$. To see that $D(A) = D(B)$, $(B - \lambda Id)D(A) = (A - \lambda Id)D(A) = X$. Thus $B - \lambda Id : D(A) \to X$ is bijective since $\lambda \in \rho(A) \cap \rho(B)$ and $B - \lambda Id : D(B) \to X$ is bijective. This completes the proof. □

Remark 1.146. Let $\gamma \in \mathbb{R}$. A semigroup $\{S(t)\}_{t\geq}$ is call γ-contractive if $\|S(t)\| \leq e^{\gamma t}$. A generalization of Hille-Yosida: if $(\gamma, \infty) \subset \rho(A)$ and $\|R_\lambda\| \leq \frac{1}{\lambda-\gamma}$ for all $\lambda > \gamma$.

Theorem 1.147. *Let $U \subset \mathbb{R}^n$ be open and bounded with smooth boundary. We set $X := L^2(U)$, $D(A) := H_0^1 \cap H^2$ and $A := -L$ where Lu is the usual elliptic operator with smooth time-independent coefficients with uniform ellipticity. Then A generates a γ-contraction semigroup $\{S(t)\}_t$ for some $\gamma \geq 0$ and given some $g \in D(A)$, $w(t) := S(t)g$ solves $\partial_t u + Lw = 0$ in $(0, \infty) \times U$ with zero boundary conditions and $w(0) = g$.*

Proof. $D(A)$ is dense in $L^2(U)$. We need to show A is closed and $(\gamma, \infty) \subset \rho(A)$ and $\|R_\lambda\| \leq \frac{1}{\lambda-\gamma}$. For closedness, suppose $\{u_k\}_k \subset D(A)$ with $u_k \to u$ and $Au_k \to f$ in L^2. By elliptic regularity:

$$\|u_k - u_l\|_{H^2} \leq C(\|Au_k - Au_l\|_L^2 + \|u_k - u_l\|_L^2).$$

Thus $\{u_k\}_k$ is Cauchy in H^2 and $u_k \to u$ in H^2 and $Au_k \to Au$ in L^2, and so $Au = f$. Now we show $(\gamma, \infty) \subset \rho(A)$ and $\|R_\lambda\| \leq \frac{1}{\lambda-\gamma}$ for $\lambda > \gamma$. Recall that there exists $\beta > 0, \gamma \geq 0$ such that

$$\beta\|u\|_{H_0^1}^2 \leq B[u,u] + \gamma\|u\|_{L^2}^2.$$

Then for this $\gamma > 0$, we have that for the boundary problem $Lu + \lambda u = f$ with zero boundary conditions has a unique weak solution (from Lax-Milgram) for each $f \in L^2$. By elliptic regularity, this solution u is in H^2, and so $u \in D(A)$. Rewrite the BVP as $\lambda u - Au = f$ so $\lambda Id - A : D(A) \to X$ is bijective. So $\lambda \in \rho(A)$ and so $[\gamma, \infty) \subset \rho(A)$.

Finally for the resolvent operators, from weak formulation

$$B[u,v] + \lambda(u,v) = (f,v)$$

with $v = u$, we have with the elliptic bound:

$$B[u,u] + \lambda\|u\|_{L^2} \leq (f,u) \leq \|f\|\|u\| \implies (\lambda - \gamma)\|u\|_[L^2] \leq \frac{1}{\lambda - \gamma}\|f\|_{L^2}$$

where $u = R_\lambda f$. This gives the operator bound for the resolvent. We simply apply Hille-Yosida to complete the proof. $\square$