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What is an ODE/PDE?

Def: Let $I \subset \mathbb{R}$ be an open interval and $m \in \mathbb{N}$. Then an m-th order ODE for and unknown function $y: I \to \mathbb{R}^n$ is of the general form

$$F(t, y(t), y'(t), \ldots, y^{(m)}(t)) = 0 \quad (\#1)$$

for some function F defined on $I \times \mathbb{R}^n \times \ldots \times \mathbb{R}^n$

Terminology:

- If F is linear in $y, y', \ldots, y^{(m)}$, then (#1) is a linear ODE,
$$\sum_{k=0}^{m} a_k(t) y^{(k)}(t) = g(t)$$
- If F does not depend on t explicitly we call the ODE autonomous
- If F is $\mathbb{R}^n$-valued, we call it a system.

a PDE is a functional equation for unknown function of more than one independent variables and its partial derivatives.

Classical Mechanics:

→ Newton's Second Law of motion: $F = m \cdot a$ (Force, mass, acceleration)

Spring: Hook's Law: $F = -Ky$, ⇒ $-Ky = ma = m \cdot \frac{d^2y}{dt^2}$ ⇒

$$\frac{d^2y}{dt^2} = -\omega^2 y, \quad \omega^2 = \frac{K}{m}$$

Harmonic Oscillator: general solution $y(t) = A\cos(\omega t) + B\sin(\omega t)$

Inverse square law for gravitational force:

$$\vec{F} = -\frac{g\, m\, M}{|\vec{y}|^2} \frac{\vec{y}}{|\vec{y}|}$$

$$\Rightarrow \quad m \cdot \frac{d^2\vec{y}}{dt^2} = -\frac{g\, m\, M}{|\vec{y}|^2} \cdot \frac{\vec{y}}{|\vec{y}|}; \quad \vec{y}: I \to \mathbb{R}^3$$

Electromagnetism: (Maxwell's)

electric field $\vec{E}: \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}^3$
magnetic field $\vec{B}: \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

ρ = charge density
j = current density

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\partial_t \vec{B}$$

$$\nabla \times \vec{B} = \mu_0 (\vec{j} + \varepsilon_0 \partial_t \vec{E})$$

Vacuum: $\rho = 0$, $j = 0$

$$\nabla \times \underbrace{(\nabla \times \vec{E})}_{= -\partial_t \vec{B}} =$$

$$-\partial_t \nabla \times \vec{B} = -\partial_t (\mu_0 \varepsilon_0 \partial_t \vec{E}) = -\mu_0 \varepsilon_0 \partial_{tt} \vec{E}$$

$$\Rightarrow \nabla \times (\nabla \times \vec{E}) = -\mu_0 \varepsilon_0 \partial_{tt} \vec{E}$$

But $\nabla \times (\nabla \times \vec{E}) = -\Delta \vec{E} + \nabla(\underbrace{\nabla \cdot \vec{E}}_{0 \text{ by Maxwell}})$

Vector Calculus

$\Rightarrow$ $\boxed{\frac{\partial^2 \vec{E}}{(\partial t)^2} - c^2 \Delta \vec{E} = 0}$ $\to$ Linear wave equation

$\Rightarrow$ Light is a wave in a vacuum

$\hookrightarrow$ Similar for $\vec{B}$.

Electrostatics: (no time dependence)

$\nabla \times E = 0$, so we can $\vec{E}$ as the gradient of a function in a simply connected domain (such as $\mathbb{R}^3$).

$$\vec{E} = -\nabla V$$ $\to$ Electrostatic potential $V: \mathbb{R}^3 \to \mathbb{R}$

Since $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$, $\Rightarrow$ $\boxed{-\Delta V = \frac{\rho}{\varepsilon_0}}$ $\to$ Poisson's Equation.

Basic Questions:

→ What does it mean to solve an ODE/PDE?
Given an ODE w/ some additional conditions, (data) such as initial conditions or boundary conditions.

(*) Existence of a solution?? (Given a set of data)

(*) Uniqueness of solution given set of data??

(*) Continuous dependence of solutions on the data

→ Well-posedness if all three hold

ODE

Picard-Lindelöf Theorem:

Goal: Prove local existence and uniqueness for initial value problems for first-order systems of ODEs of the form:

$$\begin{cases} \dfrac{dy}{dt} = F(t, y(t)) \\ y(t_0) = y_0 \end{cases}; \quad \begin{array}{l} F: I \times U \to \mathbb{R}^n \\ I \subseteq \mathbb{R} \text{ is open} \\ U \subseteq \mathbb{R}^n \text{ is open,} \\ t_0 \in I,\ y_0 \in U. \end{array}$$

→ Local existence means, under suitable assumptions on F, there exists some $\delta > 0$ such that the IVP has a differentiable solution $y: (t_0 - \delta, t_0 + \delta) \to \mathbb{R}^n$.

Remark: An m-th order ODE $x^{(m)} = H(t, x, x', \ldots, x^{(m-1)})$ for an unknown real-valued function, $x(t)$, can be recast into a system of the first order form by setting

$y_1(t) = x(t),\ y_2(t) = x'(t), \ldots, y_m(t) = x^{(m-1)}(t)$

We obtain:

$$\underbrace{\frac{dy}{dt} = \frac{d}{dt}\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \underbrace{\begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ y_{m-1} \\ f(t, y_1, y_2, \dots, y_m) \end{pmatrix}}_{F(t,y)}}_{\text{Can be solved.}}$$

<u>Lemma</u> (Integral equation for ODE): Let $I \subset \mathbb{R}$, $U \subset \mathbb{R}^n$ be open, $F: I \times U \to \mathbb{R}^n$ be continuous. A continuous function $y: J \to U$ on some open interval $J \subset I$ is a solution to the IVP:

$$\begin{cases} \frac{dy}{dt} = F(t, y(t)) \\ y(t_0) = y_0 \end{cases}$$

<u>iff</u> for all $t \in J$ $\quad y(t) = y(t_0) + \int_{t_0}^{t} F(s, y(s))\, ds$

<u>Proof</u>: ($\Rightarrow$) Integrate the ODE

($\Leftarrow$) fundamental Theorem of Calculus. ■

The idea is to build a $y(t) = y(t_0) + \int_{t_0}^{t} F(s, y(s))\, ds$.

<u>Key idea of P-L Theorem</u>: Define the map

$(Ty)(t) := y_0 + \int_{t_0}^{t} F(s, y(s))\, ds$. Then a solution to the integral equation is a fixed point for T, i.e. $y = Ty$.

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Equivalent integral formulation for ODE problem:

$y: J \to U$, $J \subseteq I$ is a solution iff

$$y(t) = \underbrace{y_0 + \int_{t_0}^{t} F(s, y(s))\, ds}_{T(y)(t)} \text{ for all } t \in J$$

Notation and review: $x = (x_1, \ldots, x_n)$; $\|x\| = \sqrt{\sum_j x_j^2}$.

Derivatives:

Def: Let $U \subset \mathbb{R}^n$ be open. Then the j^{th} partial derivative of $f: U \to \mathbb{R}$ at $x \in U$:

$$(\partial_j f)(x) = \frac{\partial f}{\partial x_j}(x) := \lim_{h \to 0} \frac{f(x + he_j) - f(x)}{h}.$$

Def: a function f from $U \to \mathbb{R}^m$ is differentiable at $x \in U$ if there exists a linear map $L_x: \mathbb{R}^n \to \mathbb{R}^m$ s.t.

$$\lim_{h \to 0} \frac{|f(x+h) - f(x) - L_x h|}{|h|} = 0.$$

We write $Df(x) = L_x$. Recall that:

$$Df(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1}(x) & \cdots & \frac{\partial f_m}{\partial x_n}(x) \end{bmatrix}.$$

Recall $f: U \to \mathbb{R}^m$ is continuously differentiable or C^1 if $Df: U \to \mathbb{R}^{m,n}$ is continuous, or equivalently, that all partial derivatives are continuous.

Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. Then, the α-th partial derivative of f is:

$$D^\alpha f = \partial^\alpha f = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3} \ldots \partial_n^{\alpha_n} f = \frac{\partial^{|\alpha|}}{\partial_{x_1} \partial_{x_2} \ldots \partial_{x_n}} f, \quad |\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n.$$

<u>Metric Spaces:</u>

<u>Def</u>: A metric space (M, d) is a set M with a distance function, or metric, $d: M \times M \to \mathbb{R}^+$ such that

(i) $d(x,y) = 0$ iff $x = y$.

(ii) $d(x,y) = d(y,x)$

(iii) $d(x,y) \leq d(x,z) + d(z,y)$, for all $x, y, z \in M$.

Remark: Every normed vector space is a metric space w/ $d(x,y) = \|x-y\|$.

Def. Let (M,d) be a metric space. A sequence of points $\{x_n\}_1^\infty \subseteq M$ <u>converges to $x \in M$</u> if

$$d(x_n, x) \to 0 \text{ as } n \to \infty.$$

A sequence is Cauchy if for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $n, m \geq N$,

$$d(x_n, x_m) \leq \varepsilon$$

Def: A metric space (X,d) is <u>complete</u> if every Cauchy sequence converges in X.

<u>Ex</u>: Let $a, b \in \mathbb{R}$ w/ $a < b$. Let $M = \mathcal{C}([a,b]; \mathbb{R}^n) = \{f: (a,b) \to \mathbb{R}^n \text{ cont}\}$ w/

$$d(f,g) = \sup_{t \in [a,b]} |f(t) - g(t)|$$

Then $(\mathcal{C}([a,b]; \mathbb{R}^n), d)$ is a complete metric space.

<u>Q</u>: What happens when $K \subset \mathbb{R}^n$, $\mathcal{C}([a,b], K)$? <u>closed subspace</u>

Def: Let (M,d) be a metric space. A map $T: M \to M$ is called a contraction mapping if there exists a number $\theta \in (0,1)$ such that for all $x,y \in M$

$$d(Tx, Ty) \leq \theta \cdot d(x,y).$$

Theorem (Contraction mapping theorem/Banach's fixed point theorem.)
Suppose (M,d) is a complete metric space and $T: M \to M$ is a contraction mapping. Then T has a unique fixed point.

Proof: For uniqueness. Suppose x,y are fixed points $Tx = x$, $Ty = y$. Then

$$0 \leq d(x,y) = d(Tx,Ty) \leq \theta d(x,y). \text{ Then}$$
$$(1-\theta)d(x,y) \leq 0. \text{ Then } d(x,y) = 0, \text{ so } x = y.$$

Now for existence. Let $x_0 \in M$. Define $x_{n+1} := Tx_n$, and consider $\{x_n\}_{\mathbb{N}}$. We show $\{x_n\}$ is Cauchy. Assuming this, then M is complete, there exists some $x \in M$ s.t. $x_n \to x$. But, by construction, $x = \lim_{n \to \infty} Tx_n = Tx$. Thus, $x = Tx$. Now we show Cauchy: For any $n \geq 0$,

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq \theta d(x_n, x_{n-1}) \leq \theta^2 d(x_{n-1}, x_{n-2})$$

$\leq \theta^n d(x_1, x_0)$. Now

$$d(x_m, x_n) \leq d(x_m, x_{m-1}) + \ldots + d(x_{n+1}, x_n)$$

$$\leq \theta^m d(x, x_0) + \ldots + \theta^n d(x, x_0) = d(x, x_0) \sum_{j=n}^{m-1} \theta^j$$

But $\theta < 1$, so $S_N := \sum_{j=1}^{N} \theta^j$ converges, and in particular $\{S_N\}_1^\infty$ is Cauchy. Thus, we only need to choose n, m large enough so that $\sum_{j=n}^{\infty} \theta^j < \varepsilon$. Thus, $\{x_n\}$ is Cauchy.

Def: Let $(V, \|\cdot\|_V)$, $(W, \|\cdot\|_W)$ be two normed vector spaces. Then the operator norm of a linear map $A: V \to W$ is defined via:

$$\|A\|_{L(V,W)} := \sup_{\substack{v \in V \\ \|v\|_V \le 1}} \|Av\|_W$$

a linear map is bounded if $\|A\|_{L(V,W)} < \infty$.

Then for $v \neq 0$, $\|Av\|_W \le \|A\|_L \|v\|$, Because:

$$\frac{\|Av\|}{\|v\|} = \Big\| A \underbrace{\frac{v}{\|v\|}}_{\le 1} \Big\| \le \|A\|. \checkmark$$

Lemma (Gronwall): Let $g: I \to [0,\infty)$ be a non negative continuous function on some interval $I \subseteq \mathbb{R}$. Assume that for some $t_0 \in I$ and for all $t \in I$,

$$g(t) \le A \cdot \left| \int_{t_0}^t g(s)\,ds \right| + B, \quad A, B \ge 0$$

Then, $g(t) \le Be^{A(t-t_0)}$

Proof: Let $t > t_0$. Define $G(t) := A\int_{t_0}^t g(s)\,ds + B$. Then, since $g(s) \ge 0$, $g(t) \le G(t)$ by assumption. Furthermore, we have:

$$\frac{dG}{dt} = A\,g(t).$$

Thus, $\frac{dG}{dt}(t) \le A\,G(t)$. Subtract:

$$\frac{dG}{dt} - AG(t) \le 0. \text{ Now multiply by } e^{-A(t-t_0)}.$$

Then $\frac{d}{dt}\left(G(t)e^{-A(t-t_0)}\right) = \frac{dG}{dt}e^{-A(t-t_0)} - Ae^{-A(t-t_0)}G(t) \le 0$

Now integrate from t_0 to s: Thus:

$$\int_{t_0}^s \frac{d}{dt}\left(G(t)e^{-A(t-t_0)}\right)dt = G(s)e^{-A(s-t_0)} - \underbrace{G(t_0)}_{B} \le 0. \text{ Therefore}$$

$$g(s) \le G(s) \le Be^{A(s-t_0)}$$

Def: Let $I \subseteq \mathbb{R}$ open and $U \subseteq \mathbb{R}^n$ open. A function $F: I \times U \to \mathbb{R}^n$ is Lipschitz continuous in the second variable if there exists some $L > 0$ such that for all $t \in I$ and for all $y_1, y_2 \in U$,

$$|F(t, y_1) - F(t, y_2)| \leq L \cdot |y_1 - y_2|.$$

Uniqueness of solutions

Lemma: Let $I \subseteq \mathbb{R}$ open and $U \subseteq \mathbb{R}^n$ open. Assume $F: I \times U \to \mathbb{R}^n$ is continuous and Lipschitz in the second variable. Let $t_0 \in I$, $y_0 \in U$, $J \subseteq I$ open with $t_0 \in J$. Suppose y_1, y_2 are two solutions to

$$\begin{cases} \frac{dy}{dt} = F(t, y) \\ y(t_0) = y_0 \end{cases}$$

Then $y_1 = y_2$ for all $t \in J$

Proof: By the integral formulation, we have for $j = 1, 2$, $t \in J$,

$$y_j(t) = y_0 + \int_{t_0}^{t} F(s, y_j(s))\,ds$$

Then $y_1 - y_2(t) = \int_{t_0}^{t} F(s, y_1(s)) - F(s, y_2(s))\,ds$. Then,

$$|y_1 - y_2(t)| \leq \int_{t_0}^{t} |F(s, y_1(s)) - F(s, y_2(s))|\,ds$$

$$\leq L \int_{t_0}^{t} |y_1(s) - y_2(s)|\,ds.$$

By Gronwall, $g(t) = |y_1(t) - y_2(t)|$ w/ $A = L$, $B = 0$, we thus conclude $0 \leq g(t) \leq 0$, so $g \equiv 0$, and $y_1(t) = y_2(t)$. ∎

[→ Observe that $z(t), z_0, y(t), y_0$ being solutions, then the same argument with produces $B = |y_0 - z_0|$, so we arrive at continuous dependence.]

Local Existence Theorem (Picard-Lindelöf): | PDE 9/1/2022

Let $I \subseteq \mathbb{R}$ be open and $U \subseteq \mathbb{R}^n$ open. Assume $F: I \times U \to \mathbb{R}^n$ is continuous and Lipschitz in the second variable. For any $(t_0, y_0) \in I \times U$, there exists a $\delta > 0$ and a unique continuously differentiable solution $y: (t_0-\delta, t_0+\delta)$ to the IVP:

$$(\#1) \quad \begin{cases} \frac{dy}{dt} = F(t,y) \\ y(t_0) = y_0 \end{cases}$$

Proof: It suffices to show there exists a continuous function $y: (t_0-\delta, t_0+\delta) \to U$ that satisfies the integral equation:

$$y(t) = y_0 + \int_{t_0}^{t} F(s, y(s))\,ds \quad \forall t \in (t_0-\delta, t_0+\delta) \ (\#2).$$

Define a map $T_y(t) := y_0 + \int_{t_0}^{t} F(s,y(s))\,ds$. We want to find a fixed point for T.
Let $r > 0$ be such that $\overline{B(y_0, r)} \subset U$. Let $\delta_0 > 0$ such that $[t_0-\delta_0, t_0+\delta_0] \subset I$. For $0 < \delta \le \delta_0$ (to be chosen later), consider

$$M := C([t_0-\delta, t_0+\delta]; \overline{B(y_0, r)}).$$

Let's equip M with $d(f,g) = \sup_{t \in [t_0-\delta_0, t_0+\delta_0]} |f(t) - g(t)|$.

Observe (M, d) is a complete metric space.

Claim: Let $\delta =$, then T is a contraction mapping on M.

Proof of claim: We show $T: M \to M$. Let $y \in M$. Consider

$T(y): [t_0-\delta, t_0+\delta]$, $\quad T(y)(t) = y_0 + \int_{t_0}^{t} F(s, y(s))\,ds$. Then

$$|T(y)(t) - y_0| \le \left| \int_{t_0}^{t} |F(s, y(s))|\,ds \right|.$$

Since F is continuous, $\sup_{t \in [t_0-\delta_0, t_0+\delta_0] \times \overline{B(y_0, r)}} |F(s,z)| =: K < \infty$.

Then $\left|\int_{t_0}^t F(s,y(s))\,ds\right| \le K \int_{t_0}^t 1\,ds \le K\cdot|t-t_0| \le K\cdot\delta$. Pick $\delta \le \frac{r}{K}$. Then, we get $K\delta \le r$. So, $T: M \to M$. Now to show T is a contraction mapping. Let $y_1, y_2 \in M$, consider

$$|T(y_1)(t) - T(y_2)(t)| \le \left|\int_{t_0}^t |F(s,y_1(s)) - F(s,y_2(s))|\,ds\right|$$
$$\le \int_{t_0}^t L\,|y_1(s) - y_2(s)|\,ds$$
$$\le L\,\underbrace{|t-t_0|}_{\le\delta}\,\sup_s |y_1(s) - y_2(s)|.$$

So: $d(T(y_1), T(y_2)) \le L\delta \cdot d(y_1, y_2)$. Pick $\delta := \min\{\delta_0, \frac{r}{K+1}, \frac{1}{2L}\}$.

Thus, T is a contraction mapping, and so by Banach's Fixed point theorem, T has a unique fixed point $\hat{y}$ s.t.

$$\hat{y}(t) = y_0 + \int_{t_0}^t F(s,\hat{y}(s))\,ds.$$

Remark: Recall that proof of Banach fixed point theorem is constructive, so we can obtain the local solution as the limit of a sequence of functions:

$$y_n(t) := y_0 + \int_{t_0}^t F(s, y_{n-1}(s))\,ds$$

Ex. $\begin{cases} \frac{dy}{dt} = ay & a\in\mathbb{R},\ y: I \to \mathbb{R}. \\ y(0) = 1 \end{cases}$

Observe that $F(y) = ay$ is autonomous.

$$y_0 = 1,\quad y_1 = 1 + \int_0^t a\cdot 1\,ds = 1 + at$$

$$y_2(t) = 1 + \int_0^t a(1+as)\,ds = 1 + at + \frac{1}{2}a^2t^2. \Rightarrow$$

$$y_n(t) = \sum_{k=0}^{n} \frac{1}{k!}(at)^k \longrightarrow e^{at}.$$

Lemma: (Continuous dependence on data)

Let $I \subset \mathbb{R}$ open, $U \subset \mathbb{R}^n$ open. Assume $F: I \times U \to \mathbb{R}^n$ is continuous and Lipschitz in second variable. Let $y(t)$ and $z(t)$ be two solutions to one constant L.

$$\frac{dy}{dt} = F(t, y(t)), \quad y_0 = y(t_0).$$

on the open $J \subset I$, $t_0 \in J$. Then:

$$|y(t) - z(t)| \leq |y(t_0) - z(t_0)| \cdot e^{L|t-t_0|}.$$

Proof: Follows from Gronwall's. Do proof.

Q Efficient ways to check if F is Lipschitz in second variable? assume F is also continuously differentiable in second variable. Let $y \in U$ s.t. $\overline{B(r,y)} \subset U$. Then $\forall\, y, z \in B(r,y)$:

For $y, z \in U$ $\quad F(t,y) - F(t,z) = \int_0^1 \underbrace{\frac{d}{d\eta}\left(F(t, z+\eta(y-z))\right)}_{= D_x F(t, z+\eta(y-z)) \cdot (y-z)} d\eta$

Thus, $|F(t,y) - F(t,z)| \leq \int_0^1 \|D_x F(t, z+\eta(y-z))\| |y-z| \, d\eta$

$$\leq \underbrace{\underbrace{\sup_{x \in \overline{B(r,y)}} \|D_x F(t,x)\|_{op}}_{\text{Finite}}}_{\text{Lipschitz constant}} |y-z| \implies F \text{ is Lipschitz in } y.$$

Def: Let $I \subseteq \mathbb{R}$ be open, $U \subseteq \mathbb{R}^n$ open. We say that $F: I \times U \to \mathbb{R}^n$ is locally Lipschitz in the second variable if, for every $(t_0, y_0) \in I \times U$, there exists neighborhoods $J_0 \subseteq I$ of t_0 and $V \subseteq U$ of y_0 so that $F: J \times V$ is Lipschitz in the second variable, i.e. there exists L_{JV} s.t. for all $t \in J$, all $y_1, y_2 \in V$, we have:

$$|F(t, y_1) - F(t, y_2)| \leq L_{JV} |y_1 - y_2|$$

Lemma: Let $I \subseteq \mathbb{R}$, $U \subseteq \mathbb{R}^n$ open. Assume $F: I \times U \to \mathbb{R}^n$ is continuously differentiable in the second variable. Then F is locally Lipschitz in the second variable.

Proof: See earlier.

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Lemma: (Uniqueness under local Lipschitz)
Let $I \subseteq \mathbb{R}$ be open and $U \subseteq \mathbb{R}^n$. Assume $F: I \times U \to \mathbb{R}^n$ is continuous and locally Lipschitz in the second variable. Let $t_0 \in J$ and suppose y_1, y_2 are two solutions to $y' = F(t,y)$ on some $J \subset I$ with $t_0 \in J$. If $y_1(t_0) = y_2(t_0)$, then $y_1(t) = y_2(t)$ $\forall t \in J$. ↳ interval.

Proof: Define $J' := \{t : y_1(t) = y_2(t)\}$. Since y_1, y_2 are continuous J' is closed. Now we want to show that J' is open in the subspace topology. Let $\tilde{t} \in J'$ and $\tilde{y} := y_1(\tilde{t}) = y_2(\tilde{t})$. Then by local Lipschitz, there exists open neighborhoods $\tilde{J} \subset I$ of $\tilde{t}$ and $\tilde{V} \subset U$ of $\tilde{y}$, where $F|_{\tilde{J} \times \tilde{V}}$ is Lipschitz in the second variable. Then, we have that by Grönwall's lemma that $y_1(t) = y_2(t)$ $\forall t \in \tilde{J} \cap J$. Thus,

$\tilde{J} \cap J \subset J'$, and so J' must be open. ▨

↳ Open in J.

↳ Get this far from by local Lipschitz; y_1, y_2 are solutions on $\tilde{J}$.

The Maximal interval of Existence:

Lemma: Let $I \subseteq \mathbb{R}$ open, $U \subseteq \mathbb{R}^n$ open. Assume F is continuous and locally Lipschitz in the second variable. Then, for any t_0 and y_0, there is a maximal interval of existence, $J \subseteq I$ in which the IVP.

$$\begin{cases} \dfrac{dy}{dt} = F(t,y) \\ y(t_0) = y_0 \end{cases}$$

has a unique solution, i.e. if $z: J' \to U$ is a solution, then $J' \subset J$ and $z(t) = y(t)$ for all $t \in J'$, and J must be open.

Proof: Let J be the union of all intervals, $I_\alpha \subset I$, $t_0 \in I_\alpha$ on which the IVP has a solution $y_\alpha : I_\alpha \to U$. For $t \in J$ choose some I_α with $t \in I_\alpha$ and set $y(t) := y_\alpha(t)$.
If I_β is another interval with $t \in I_\beta$ then $[t_0, t] \subseteq I_\beta \cap I_\alpha$. By uniqueness lemma, we get $y_\alpha(t) = y_\beta(t)$.
To show J is open, suppose J is of the form $(a,b] \subseteq I$.
Now consider an IVP w/ $y(b) = y_0$. Then we can find and interval around b. But then, we have two solutions on the intersecting interval, and thus by uniqueness, we may extend $(a,b]$ to $(a, b+\delta)$. ▨

<u>Proposition:</u> Let $I = (A,B) \subset \mathbb{R}$ for some $-\infty \leq A < B \leq \infty$, $\mathcal{U} \subseteq \mathbb{R}^n$ open. Assume $F: I \times \mathcal{U} \to \mathbb{R}^n$ is continuous and locally Lipschitz in the second variable. For some (t_0, y_0), let $y: \underset{\hookrightarrow \text{maximal interval}}{(a,b)} \to \mathcal{U}$ be a maximal solution to

$$\begin{cases} \frac{dy}{dt} = F(t,y) \\ y(t_0) = c_0 \end{cases}$$

If b does not coincide with B, then for every compact subset $K \subset \mathcal{U}$ and $a < c < b$, there exists time $t \in (c, b)$ such that $y(t) \notin K$. In other words, if $b < B$, then either $|y(t)|$ becomes unbounded or $y(t)$ approaches the boundary of $\mathcal{U}$ as $t \to b$, and the analogous statement for $A < a$.

<u>Proof:</u> Consider $b < B$. Let $K \subseteq \mathbb{R}^n$ be compact, $a < c < b$. Suppose $y(t) \in K$ for all $t \in (c,b)$. We claim that $y(t)$ has a continuous extension to $(a, b]$ (which would contradict maximality). We show that $y(t)$ is uniformly continuous on (c, b), which follows from:

$$|y(t_1) - y(t_2)| = \left| \int_{t_2}^{t_1} \frac{d}{dt} y(t)\, dt \right|$$

$$\leq |t_1 - t_2| \cdot \underbrace{\sup_{(s,x) \in [c,b] \times K} |F(s,x)|}_{< \infty}.$$

Thus, $y(t)$ is actually Lipschitz, so $\lim_{t \to b} y(t)$ exists, and we may extend our maximal interval to $(a, b]$ which is a contradiction. ∎

<u>Corollary:</u> Let $I \subseteq \mathbb{R}$ open. Assume $F: I \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and locally Lipschitz in the second variable. Suppose there exists continuous functions $p, q: I \to [0, \infty)$ such that

$$|F(t,x)| \leq p(t) \cdot |x| + q(t)$$

for all $t \in I$ and $x \in \mathbb{R}^n$. Then for every $(t_0, y_0) \in I \times \mathbb{R}^n$, the IVP

$$\begin{cases} \frac{dy}{dt} = F(t,y), \quad y(t_0) = c_0 \end{cases}$$

has a unique solution on all of I.

Proof: Let $y: J \to \mathbb{R}^n$ be the maximal solution to the IVP. Suppose $J=(a,b)$ with b not equal to the end point of I. Then, $\limsup |y(t)| = \infty$. On the other hand,

$$|y(t)| \leq |y(t_0)| + \left|\int_{t}^{t_0} F(s, y(s))\,ds\right|$$

$$\leq |y(t_0)| + \left|\int_{t}^{t_0} p(s)|y(s)| + g(s)\,ds\right|.$$

$$\leq \underbrace{|y(t_0)| + \int_{t_0}^{b} g(s)\,ds}_{\text{constant}} + \int_{t_0}^{t} \underbrace{p(s)}_{\substack{\text{bounded} \\ \text{on } [t_0,b]}} |y(s)|\,ds.$$

Use Gronwall to show $\limsup |y(t)| < \infty$, contradiction ∎

Maximal integral curves for autonomous ODEs.

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$$\begin{cases} \frac{dy}{dt} = F(y) \to \text{vector field independent of } t. \\ y(t_0) = y_0 \end{cases}$$

Remark on time translations:
Let $U \subseteq \mathbb{R}^n$ open and $F: U \to \mathbb{R}^n$ locally Lipschitz. Let $y: J \to U$ be a maximal integral curve for $\frac{dy}{dt} = F(y)$. Then

(1) For all $c \in \mathbb{R}$,

$$y_c : J + c \to U, \quad y_c(t) := y(t-c)$$

is also a maximal integral curve for $\frac{dy}{dt} = F(y)$.

(2) If $z: J' \to U$ is a maximal integral curve with $z(s) = y(r)$, $s \in J'$, $r \in J$, then $J' = J + (s-r)$ and $z(t) = y_{s-r}(t)$.

Proof: (1) $\frac{dy_c}{dt} = \frac{dy}{dt}(t-c) = F(y(t-c)) = F(y_c(t))$.

To see maximality, simply show that another solution on $J+c$, if larger, then $-c$ would would be larger than J.

(2) From $z(s) = y(r) = y(s-(s-r)) = y_{s-r}(s)$ for our fixed s, r values. This can be treated as an initial condition, and so since both z and y are maximal solutions, $z(t) = y_{s-r}(t)$ $\forall t \in J'$.

Note: while $y(t)$ and $y_c(t)$ are different functions, their images are the same.

From (2) we obtain that U is a disjoint union of the images of maximal integral curves of $\frac{dy}{dt} = F(y)$.

Prop: Let $U \subseteq \mathbb{R}^n$ open and $F: U \to \mathbb{R}^n$, Lipschitz in second variable. For every point in U, there is a unique maximal integral curve $y: J \to U$ passing through it and exactly one of the following holds:

(1) There is a $t_0 \in J$ with $\frac{dy}{dt}(t_0) = 0$. Then $J = \mathbb{R}$ and $y(t)$ is constant. $\forall t$

(2) For all $t \in J$, $\frac{dy}{dt}(t) \neq 0$, and there exists

$r, s \in J$ with $y(r) = y(s)$. Then $J = \mathbb{R}$ and $y(t)$ is periodic

(3) For all $t \in J$, $\frac{dy}{dt} \neq 0$, and y is injective.

Proof: (1) $0 = \frac{dy}{dt}(t_0) = F(y(t_0))$. Take $y(t) = y_0$. This a solution

and so is unique.

(2) $J = J_{s-r} \Rightarrow J = \mathbb{R}$.

(3) Follows when the other two fail.

Ex: $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix} + (1-x^2-y^2)\begin{pmatrix} x \\ y \end{pmatrix} := F\begin{pmatrix} x \\ y \end{pmatrix}$.

Then, $F(x,y) = (0,0) \Leftrightarrow (x,y) = (0,0)$ (can check).

On $\mathbb{R}^2 \setminus \{0\}$, use polar coordinates:

$$x(t) = r(t)\cos(\theta(t)); \quad y(t) = r(t)\sin(\theta(t)).$$

Then, $\frac{dr}{dt} = r(1-r^2)$; $\frac{d\theta}{dt} = 1 \Rightarrow \theta(t) = t$.

Linear Systems of first-order ODEs:

Now consider linear first order systems of ODEs,

$\frac{dy}{dt} = A(t)\cdot y + b(t)$, with $A: I \to \mathbb{R}^{n\times n}$, $b: I \to \mathbb{R}^n$.
↳ Need A to be continuous.

If $b \equiv 0$, we consider what's called homogeneous.
If $b \neq 0$, we consider the non homogeneous.

Prop: Let $I \subseteq \mathbb{R}$ open, $A: I \to \mathbb{R}^{n\times n}$, $b: I \to \mathbb{R}^n$ be continuous. Then for any $(t_0, y_0) \in I \times \mathbb{R}^n$, the IVP:

$$\begin{cases} \frac{dy}{dt} = A(t)\, y + b(t) \\ y(t_0) = y_0 \end{cases}$$

has a unique solution defined on all I.

Proof: (*) Do later.

Lemma: (i) The set $\mathcal{L}$ of all solutions defined on I to the homogeneous ODE is an n-dimensional vector space over $\mathbb{R}$.

(ii) n solutions $y_1, \dots, y_n$ form a basis of the vector space of $\mathcal{L}$ if the vectors $y^1(t), \dots, y^n(t)$ form at least for 1 $t \in I$ a basis of $\mathbb{R}^n$.

Proof:

(i) By linearity of $\frac{dy}{dt} = A(t)y$, a linear combination of solutions is again a solution. (ii) For any $t_0 \in I$, the map from $\mathcal{S} \to \mathbb{R}^n$, $y(t_0)$, is an isomorphism of vector spaces. Injection comes from uniqueness, and surjection comes from existence.

Def: A set of solutions to the homogeneous linear system

$$\frac{dy}{dt} = A(t)y$$

is called a fundamental set of solutions if the set is a basis of $\mathcal{S}$.

If we collect a fundamental set of solutions $y_1, \dots, y_n : I \to \mathbb{R}^n$ as the columns of a matrix, then we obtain a map:

$$\Phi : (y_1, \dots, y_n) : I \to \mathbb{R}^n$$

Then, $\frac{d}{dt}\Phi = A(t)\Phi$. The map Φ is called a fundamental matrix for the linear system.

Proposition: (Liouville's Formula)

Let $I \subset \mathbb{R}$ open, and $A : I \to \mathbb{R}^{n \times n}$ be continuous. Let $\Phi : I \to \mathbb{R}^{n \times n}$ be a fundamental matrix for $\frac{dy}{dt} = A(t)y$. Then

$$\frac{d}{dt}(\det(\Phi)) = \operatorname{tr}(A(t)) \det(\Phi),$$

i.e. for any $t_0 \in I$, $\det(\Phi(t)) = \det(\Phi(t_0)) \exp\left(\int_{t_0}^{t} \operatorname{tr} A(s)\, ds\right)$.

Proof: Follows by integrating factors.

in preparation:

$$\frac{d}{dt}(\det(\Phi(t)))=\sum_{j=1}^{n}\det(v_1,\ldots,v_{j-1},\frac{dv_j}{dt},v_{j+1}),$$

where $v_1(t),\ldots,v_n(t)$ are columns of $\Phi(t)$. $\Phi(t)=(v_1(t),\ldots,v_n(t))$

$$\frac{1}{h}\left(\det\Phi(t+h)-\det(\Phi(t))\right)=\underset{\substack{\uparrow\\ \text{by linearity of determinant in every column}}}{\sum_{j=1,n}}\det\left(v_1(t),\ldots,v_{j-1}(t),\frac{v_j(t+h)-v_j(t)}{h},v_{j+1}(t+h),\ldots,v_n(t+h)\right).$$

Let $h\to 0$, and this shows the claim.

Proof of formula: We first show that the formula holds for all $t\in I$ which $\Phi(\hat{t}):=Id$.

Then by previous computation,

$$\frac{d}{dt}(\det(\Phi(\hat{t})))=\sum_{j=1}^{n}\det(e_1,\ldots,e_{j-1},\underbrace{\frac{dv_j}{dt}}_{\substack{=A(\hat{t})v_j(\hat{t})\\ =A(\hat{t})e_j}},e_{j+1},\ldots,e_n)$$

$$=\sum_{j=1}^{n}A_{jj}(\hat{t})$$

$$=tr(A(\hat{t})).$$

Now the general case: Let $\Phi(\hat{t})$ be arbitrary, but invertible b.c. $\Phi(t)$ is fundamental. $\longrightarrow \psi(\hat{t})=Id$.

Define $\Psi(t):=\Phi(t)\,\Phi^{-1}(\hat{t})$. Then by previous step,

$$\frac{d}{dt}\det(\Psi(t))=tr(A(\hat{t}))\det(\Psi(\hat{t}))$$

$$\Rightarrow \frac{d}{dt}\left(\det(\Phi(t))\det(\Phi(\hat{t}))^{-1}\right)=tr(A(\hat{t}))\cdot\det(\Phi)\det(\Phi(\hat{t}))^{-1}$$

$\Rightarrow$ completes the computation.

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Linear Systems w/ constant Coefficients:

Let $A \in \mathbb{R}^{n \times n}$. Consider

$$\begin{cases} \frac{dy}{dt} = Ay \\ y(0) = y_0 \end{cases} \rightarrow \text{Global solutions exist.}$$

If $n=1$: $y(t) = y_0 e^{at}$.

$n \geq 2$: $y(t) = y_0 e^{At}$
define this & ensure it holds "nicely"

Define: $e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!}$

Recall: We equip the vector space $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ with the operator norm:

$$\|A\|_{op} := \sup_{\|v\| \leq 1} \|Av\|$$

Then $(\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n), \|\cdot\|)$ is a Banach space.

Lemma: For $A, B \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, $x \in \mathbb{R}^n$,

(i) $|Ax| \leq \|A\| \, |x|$
(ii) $\|AB\| \leq \|A\| \, \|B\|$
(iii) $\|A^k\| \leq \|A\|^k$, $k \in \mathbb{N}$.

Proof: (i) For $x \neq 0$, $|Ax| = |A(\frac{x}{|x|})| \cdot |x| \leq \|A\| \cdot |x|$.
(ii) $|ABx| \leq \|A\| \cdot |Bx| \leq \|A\| \cdot \|B\| \cdot |x|$. $x \neq 0$. Then
$\|AB\| \leq \|A\| \cdot \|B\|$.
(iii) Follows from (ii).

Prop: Given $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, $t > 0$, then

$$e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}$$

as a function on $[-T, T] \rightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ converges absolutely and uniformly

Proof: By the preceding lemma;

$$\left\| \frac{(At)^k}{k!} \right\| \leq \frac{\|A\|^k \cdot t^k}{k!}.$$

Hence for all time $|t| \leq T$, $\sum_{k=0}^{\infty} \left\| \frac{(At)^k}{k!} \right\| \leq \sum_{k=0}^{\infty} \frac{\|A\|^k |t|^k}{k!} \leq e^{\|A\| |t|} \leq e^{\|A\| T} < \infty$

Thus by the Weierstrass M-test, the convergent statements follow. ∎

$$\frac{dy}{dt} = Ay.$$

Lemma: If $A, B \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ commute, then

$$e^{A+B} = e^A e^B$$

Proof: $e^{A+B} = \sum_{k=0}^{\infty} \frac{(A+B)^k}{k!}$, $(A+B)^k =$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} k! \sum_{l=0}^{k} \frac{A^l B^{k-l}}{l!(k-l)!} \text{ Commute}$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{A^l B^{k-l}}{l!(k-l)!} = \left(\sum_{l=0}^{\infty} \frac{A^l}{l!} \right) \left(\sum_{k=0}^{\infty} \frac{B^k}{k!} \right)$$

→ Product of 2 absolutely convergent series is given by its Cauchy product.
↳ See Rudin Theorem 3.50. ∎

Corollary: $(e^A)^{-1} = e^{-A}$, b.c. $e^{-A} e^A = e^0 = Id$. ∎

Lemma: Let $A \in \mathbb{R}^{n \times n}$. Then

$$\frac{d}{dt}(e^{At}) = Ae^{At}.$$

Proof: For some fixed $t \in \mathbb{R}$, $\lim_{h \to 0} \frac{1}{h}\left(e^{A(t+h)} - e^{At}\right) =$

$$\lim_{h \to 0} \frac{1}{h}\left(e^{Ah} e^{At} - e^{At}\right) = \lim_{h \to 0} \left(\frac{e^{Ah} - Id}{h}\right) e^{At}$$

$$= \lim_{h \to 0} \frac{1}{h}\left(Ah + R(h)\right) e^{At}$$

$$= \lim_{h \to 0}\left(A + \frac{R(h)}{h}\right) e^{At}$$ Since $\|R(h)\| \le \sum_{k=2}^{\infty} \frac{\|(Ah)^k\|}{k!} \le h^2 \sum_{k=2}^{\infty} \frac{\|A\|^k |h|^{k-2}}{k!}$

$\le |h|^2 e^{\|A\|}$.

Thus, $R(h) \to 0$ faster than h. Hence

$$= \lim_{h \to 0}\left(A + \frac{R(h)}{h}\right) e^{At} = Ae^{At} \quad \blacksquare$$

Thus: $\frac{d}{dt}\left(e^{At} y_0\right) = Ae^{At} y_0$. $t = 0 \Rightarrow e^{At} = Id$, so $e^{At} y_0 = y_0$.

Prop: Let $A \in \mathbb{R}^{n \times n}$ and $y_0 \in \mathbb{R}^n$. Then the solution to the IVP

$$\begin{cases} \frac{dy}{dt} = Ay \\ y(0) = y_0 \end{cases}$$

is given by $y(t) = e^{At} y_0$.

If $\{v_1, \ldots, v_n\}$ is a basis for $\mathbb{R}^n$, then the collection $\{v_1 e^{At}, \ldots, v_n e^{At}\}$ is a basis for $\mathcal{L}$ (the vector space of solutions to $\frac{dy}{dt} = Ay$). In particular, the columns of e^{At} form a fundamental set of solutions, i.e., e^{At} is a fundamental matrix for our system.
↳ Take $v_j = e_j$.

Q: How to compute e^{At} for some A?

→ If diagonal: $\Rightarrow \exp\left(\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} t\right) = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} \lambda_1 t & & \\ & \ddots & \\ & & \lambda_n t \end{bmatrix}^k$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} (\lambda_1 t)^k & & \\ & \ddots & \\ & & (\lambda_n t)^k \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}.$$

Lemma: If $A, B \in \mathbb{R}^n$ and $A = P^{-1}BP$ for $P \in GL(n)$, then

$$e^A = P^{-1}e^B P.$$

Proof: $e^A = \sum_{k=0}^{\infty} \frac{(P^{-1}BP)^k}{k!} = \sum_{k=0}^{\infty} \frac{P^{-1}B^kP}{k!} = \lim_{n\to\infty} P^{-1}\left(\sum_{k=0}^{n} \frac{B^k}{k!}\right)P$

$= P^{-1}(e^B)P.$ ■

Corollary: If $A = P^{-1}DP$, for $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$, $P \in GL(n)$, then
$$e^A = P^{-1}\begin{bmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{bmatrix}P.$$

→ If $A \in \mathbb{R}^{n\times n}$ is symmetric, then spectral theory applies, and we are happy.

Lemma: If $v \in \mathbb{R}^n$ is an eigenvector of $A \in \mathbb{R}^{n\times n}$, then

$$y_v : \mathbb{R} \to \mathbb{R}^n, \quad y_v(t) = e^{\lambda t}v$$

is a solution to the IVP:

$e^{At}v = e^{\lambda t}v$
↳ can show directly

$$\frac{dy}{dt} = Ay$$
$$y_0 = v.$$

If $v_1, \ldots, v_n \in \mathbb{R}^n$ are n linearly independent eigenvectors of A with eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, then
$y_{v_1}(t) = e^{\lambda_1 t}v_1, \ldots, y_{v_n}(t) = e^{\lambda_n t}v_n$ form a fundamental set of solutions.

Moreover, $e^{At} = P\begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}P^{-1}$ w/
$P = [v_1 | \ldots | v_n]$.

If v is an eigenvector of $A \in \mathbb{R}^{n\times n}$ with eigenvalue λ, then

$$e^{At}v = e^{\lambda Id t} e^{(A-\lambda Id)t} v$$

$$= e^{\lambda Id t} \cdot \sum_{k=0}^{\infty} \frac{t^k}{k!} \underbrace{(A-\lambda Id)^k v}_{0 \text{ for } k \geq 1}$$

$$= e^{\lambda Id t} \cdot Id\, v \Rightarrow$$

$$e^{At}v = e^{\lambda t} v.$$

Def: We call $v \in \mathbb{R}^n$ a generalized eigenvector of A for eigenvalue λ if there exists some $m \in \mathbb{N}$ s.t.

$$(A - \lambda Id)^m v = 0.$$

For generalized eigenvector

$$e^{At}v = e^{\lambda t} \cdot \sum_{k=0}^{\infty} \frac{t^k}{k!} (A - Id\lambda)^k v$$

$$= e^{\lambda t} \cdot \sum_{k=0}^{m-1} \frac{t^k}{k!} (A - Id)^k v, \text{ b.c. } (A-\lambda Id)^k v = 0 \text{ for } k \geq 0.$$

Q: How compute explicitly e^{At}?
How to obtain an explicit fundamental set of solutions for 1st order ODE system?

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(*) If $A, B \in \mathbb{R}^{n\times n}$ with $A = P^{-1}BP$, for $P \in GL(n)$, then

$$e^A = P^{-1} e^B P.$$

(*) If $v \in \mathbb{R}^n$ is an eigenvector of $A \in \mathbb{R}^{n\times n}$ with eigenvalue λ, Then

$$e^{At} v = e^{\lambda t} v.$$

Proof: $e^{At} v = e^{(\lambda Id)t} e^{(A - \lambda Id)t} v$

$$= e^{(\lambda Id)t} \Big(\sum_{k=0}^{\infty} \frac{t^k}{k!} (A - \lambda Id)^k v \Big) \quad \to 0 \text{ for } k \neq 0$$

$$= e^{\lambda t} Id\, v \qquad \longrightarrow \quad e^{(\lambda Id)t} = e^{\lambda t} \cdot Id$$

$$= e^{\lambda t} v.$$

Def: Let $A \in \mathbb{C}^{n\times n}$ and let $\lambda \in \mathbb{C}$ be an eigenvalue of A. Then a vector is a generalized eigenvector of A for the eigenvalue λ if there exists $r \in \mathbb{N}$ s.t.

$$(A - \lambda Id)^r v = 0.$$

Then $e^{At} v = e^{\lambda t} \cdot \Big(\sum_{k=0}^{\infty} \frac{t^k}{k!} (A - \lambda Id)^k v \Big).$

Theorem: For every $A \in \mathbb{C}^{n\times n}$, there exists a basis of $\mathbb{C}^n$ which consists of generalized eigenvectors. For every eigenvalue of A of multiplicity m, this basis contains m generalized eigenvectors for the eigenvalue λ.

Proof: See [Hirsch-Smale, App III].

If all eigenvalues of $A \in \mathbb{R}^{n\times n}$ are real, we can compute efficiently an explicit fundamental set of solutions

Q: What if $A \in \mathbb{R}^{n\times n}$ has complex eigenvalues?

Lemma: Assume $A \in \mathbb{R}^{n\times n}$ has complex eigenvalue $\lambda = a + ib$, $b \neq 0$, with eigenvector $w = u + iv$, $u, v \in \mathbb{R}^n$. Then, $\bar{\lambda}$ is also an eigenvalue of A with eigenvector $\bar{w}$. Moreover,

$$y_2(t) := \mathrm{Re}(e^{\lambda t} w) = e^{at}(\cos(bt)u - \sin(bt)v)$$
$$y_1(t) := \mathrm{Im}(e^{\lambda t} w) = e^{at}(\sin(bt)u + \cos(bt)v)$$

are 2 linearly independent solutions to
$$\frac{dy}{dt} = Ay$$

Proof: Since A is a real matrix,

$$Aw = \lambda w \Rightarrow \overline{Aw} = \overline{\lambda w} \Leftrightarrow A\bar{w} = \bar{\lambda}\bar{w}.$$

Moreover,
$$e^{\lambda t} w = e^{(a+ib)t}(u+iv)$$
$$= e^{at}(\cos(bt) + i\sin(bt))(u+iv)$$
$$= e^{at}(\cos(bt)u - \sin(bt)v) + ie^{at}(\sin(bt)u + \cos(bt)v)$$

Assume $c_1 y_2(t) + c_2 y_1(t) = 0$. So with $t = 0$,

$$c_1 u + c_2 v = 0.$$

at $t = \frac{\pi}{2b}$, $\cancel{e^{\frac{a\pi}{2b}}}(-c_1 v + c_2 u) = 0 \Rightarrow c_1 = c_2 = 0.$

Thus, y_2, y_1 are linearly independent. If

$\frac{dy}{dt} = Ay$, then $\frac{d}{dt}(\mathrm{Re}(y)) = A\,\mathrm{Re}(y)$

$$\frac{d}{dt}(\mathrm{Re}(y) + i\,\mathrm{Im}(y)) = A(\mathrm{Re}(y) + i\,\mathrm{Im}(y)) \Rightarrow$$

$$\frac{d}{dt}(\mathrm{Re}(y)) + i\frac{d}{dt}(\mathrm{Im}(y)) = A(\mathrm{Re}(y)) + iA\,\mathrm{Re}(y). \Rightarrow$$

Real and imaginary part are real-valued solutions.

Q: What if A has a complex eigenvalue with geo multiplicity $<$ alg multiplicity?

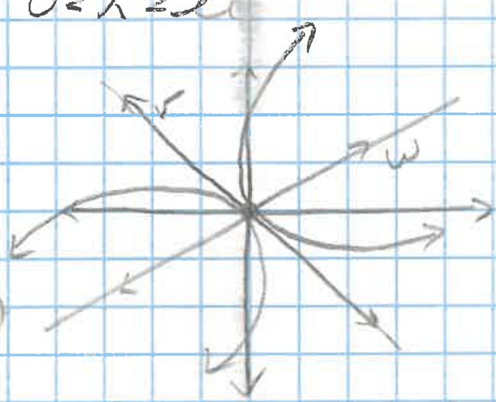
The case $n=2$: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

① Two distinct real eigenvalues: λ, μ, eigenvectors v, w. Then:

General solution:

$$y(t) = c_1 e^{\lambda t} v + c_2 e^{\mu t} w.$$

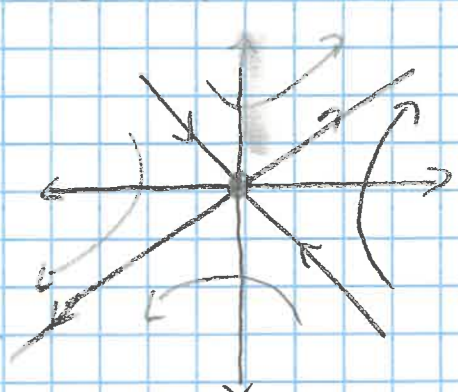
$0 < \lambda < \mu$



w/ $\mu > \lambda > 0$.

Source:

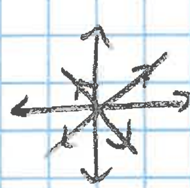
$\lambda < 0 < \mu$



Saddle:

② One repeated:

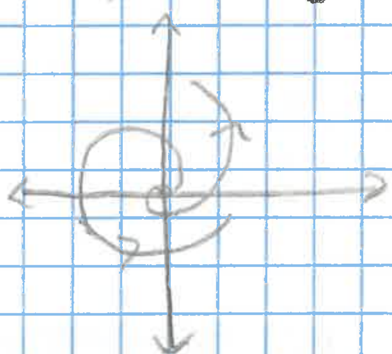
⊛ geom. multiplicity 2:



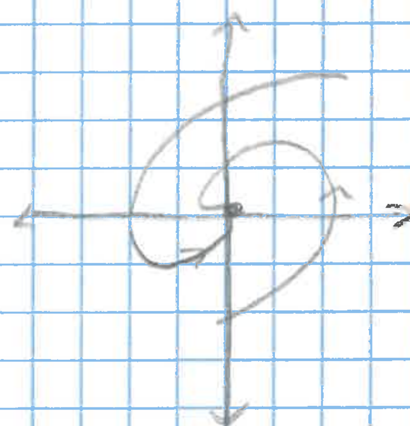
(*) geom. multiplicity = 1

$y(t) = c e^{\lambda t} + c_2 t e^{\lambda t} v.$

(3) Complex eigenvectors $\lambda = a + ib$, $\bar{\lambda} = a - ib$



$a < 0$



$a > 0$

$a = 0$

$y(t) = c_1 e^{at} \cos(bt) + c_2 e^{at} \sin(bt).$

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Nonhomogeneous Linear Systems:

Consider $\frac{dy}{dt} = A(t)y + b(t)$; $A(t) \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ } continuous (#)

$y(0) = y_0$

Observation: Let $y^1(t), y^2(t)$ be two solutions to the ODE. Then, $z(t) = y_1(t) - y_2(t)$ is a solution to the homogeneous equation

$$\frac{dz}{dt} = A(t)z.$$

→ Assume y_p satisfies $\frac{dy_p}{dt} = A(t)y_p + b(t)$. We need to satisfy initial condition y_0. Then we solve

$$\frac{dw}{dt} = A(t)w, \quad w(0) = y_0 - y_p(0)$$

Then, $\frac{d}{dt}(w + y_p) = A(t)w + A(t)y_p + b(t) = A(t)[w + y_p] + b(t)$,

and $w(0) + y_p(0) = y_0$, and so is a solution.

⇒ Thus to solve any IVP to (#), we need to exhibit just one particular solution.

Prop: Let $\Phi(t)$ be a fundamental matrix solution for $\frac{dy}{dt} = A(t)y$, and let $t_0 \in I$. Then,

$$y_p(t) = \Phi(t)\left(\int_{t_0}^{t} \Phi^{-1}(s)\, b(s)\, ds\right)$$

is a particular solution to (#).

Proof: $\frac{d}{dt} y_p(t) = \frac{d\Phi}{dt}\left(\int_{t_0}^{t} \Phi^{-1}(s) b(s)\, ds\right) + \Phi^{-1}(t) b(t)\, \Phi(t)$

$$= \frac{d\Phi}{dt}\left(\int_{t_0}^{t} \Phi^{-1}(s) b(s)\, ds\right) + b(t)$$

$\dot{\Phi}(t) = A(t)\Phi(t)$

$$= \underbrace{A(t)\,\Phi(t)\int_{t_0}^{t} \Phi^{-1}(s) b(s)\, ds}_{y_p} + b(t) = A(t)y_p + b(t).$$

Vector fields and flows:

Let $U \subseteq \mathbb{R}^n$ be open,
→ Let $F: U \to \mathbb{R}^n$ be a $C^1(U)$ vector field.

Consider the autonomous ODE:

$$\frac{dy}{dt} = F(y)$$

Introduce flow associated with autonomous ODE:

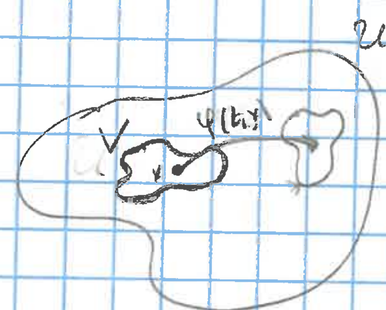
$$\varphi: I \times V \longrightarrow U$$
$$(t,x) \longmapsto \varphi(t,x)$$

where for every fixed point $x \in V$,

$$t \longmapsto \varphi(t,x)$$

solves the IVP:

$$\begin{cases} \dfrac{dy}{dt} = F(y) \\ y(0) = x \end{cases}$$



Lemma: Let $y:[a,b] \longrightarrow U$ be a solution to the autonomous ODE. Then there exists $r, L > 0$ s.t.

a) $\forall x \in \overline{B(x_0,r)}$, $x_0 = y(a)$, there exists a solution $z(t)$ to the autonomous ODE w/ $z(t):[a,b] \to U$ with $z(a) = x$.

b) For two integral curves $y_1(t), y_2(t)$ with $y_1(a), y_2(a) \in \overline{B(x_0,r)}$,

$$|y_1(t) - y_2(t)| \le |y_1(a) - y_2(a)| e^{Lt}; \quad a \le t \le b.$$

Proof: Choose $\rho > 0$ small enough so that $\underbrace{\bigcup_{t\in[a,b]} \overline{B(y(t),\rho)}}_{= K} \subset U$.
Then, we define

$$L := \max_{x \in K} \|DF(x)\|_{op} < \infty.$$

We set $r :=$

• Let $x \in \overline{B(x_0,r)}$ and denote $z:(\alpha,\beta) \to U$ the maximal

integral curve with $z(0)=x$, with $\alpha<0<\beta$.
We claim
$$d(t) := |z(t)-y(t)| < \rho \text{ for all } 0 \le t \le \min\{\beta, b\}.$$

Proof of claim: Suppose not. Then there is a $t^* \le \min\{b,\beta\}$ such that $d(t) < \rho$ for $t<t^*$, but $d(t^*) \ge \rho$. From the integral equations:
$$y(t) = \overset{x_0}{y(0)} + \int_0^t F(y(s))\,ds.$$
$$z(t) = \underbrace{z(0)}_{x} + \int_0^t F(z(s))\,ds.$$
We get
$$|z(t)-y(t)| \le |x_0 - x| + \int_0^t |F(y(s)) - F(z(s))|\,ds$$
$$\le r + \overbrace{\max_{x\in K} \|DF(x)\|}^{L} \int_0^t d(s)\,ds.$$
Sup to t^*, $z(s), y(s) \in K$.

By Gronwall:
$$|z(t)-y(t)| \le r\cdot e^{Lt} \text{ for } 0 \le t \le t^*.$$
$$= r\, e^{Lb} \underbrace{e^{L(t-b)}}_{\le e^{L(t^*-b)} \le 1}$$
Choose $r := \rho e^{-Lb}$, then $d(t^*) < \rho \Rightarrow C$.

Now we conclude $\beta > b$. If $\beta < \infty$, then there is a t s.t. $z(t) \notin K$ by our characterization. But $z(t) \in K$ for $0 \le t < \min\{\beta, b\}$. Thus, $\beta > b$.

A routine Gronwall argument gives 2. ■

Def: Let $V \subset U$ be a subset of U and I some interval containing 0. The (local) flow on $I \times V$ generated by $F: U \to \mathbb{R}^k$ is the uniquely determined map, if it exists
$$\varphi: I \times V \to U$$
such that

i) The partial derivative of φ w.r.t. t exists and
$$\partial_t \varphi(t,x) = F(\varphi(t,x)), \quad \varphi(0,x) = x.$$

A vector field $F: U \to \mathbb{R}^n$ does not necessarily generate a flow on any given $I \times U$.

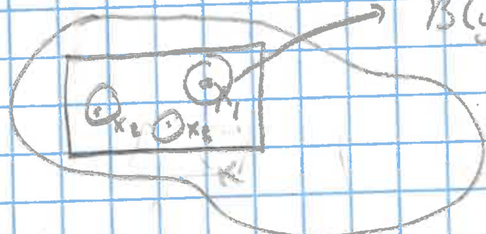
A flow on $\mathbb{R} \times U$ is called a global flow.

Ex: Let $A \in \mathbb{R}^{n \times n}$ and set $F: \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto Ax$, we get a global flow:

$$\varphi: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$$
$$(t, x) \mapsto e^{At}x.$$

Lemma: For every compact subset $K \subseteq U$, there exists an interval $[0, c]$, $c > 0$ such that F generates a local flow on $[0, c] \times K$.

Proof:



$\overline{B(y_1(0), r_1)}$; $y_1: [0, b_1] \to U$.

K is compact, so

$\overline{B(y_1(0), r_1)}$; $[0, b_1]$

$\overline{B(y_N(0), r_N)}$; $[0, b_N]$

$c := \min\{b_j\}$.

Given a flow $\varphi: I \times V \to U$, for every $t \in I$, we obtain a map $\varphi_t: V \to U$, $x \mapsto \varphi_t(x) = \varphi(t, x)$.

For $s, t \in I$ w/ $s + t \in I$, $x \in V$,

$$\varphi_s(\varphi_t(x)) = \varphi_{s+t}(x).$$

Theorem: Let $U \subseteq \mathbb{R}^n$ be open and $F: U \to \mathbb{R}^n$ be C^1. Let $V \subset U$ open and let $\varphi: I \times V \to U$ be a flow. Then for every $t \in I$ the map $\varphi_t: V \to \varphi_t(V)$ is a diffeomorphism, and its derivative $D\varphi_t$ at $x \in V$ is the value $X(t)$ of the solution to the linear matrix IVP:

$$\left\{ \frac{d}{dt} X = A(t) X, \quad X(0) = Id \right. \to \text{Variational Equation}$$

$A(t) = DF(\varphi_t(x))$

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Vectors and Flows (Cont.)

Let $U \subseteq \mathbb{R}^n$ open and $F: U \to \mathbb{R}^n$ a C^1 vector field.
For $V \subseteq U$ and $I \subseteq \mathbb{R}$ an interval with $0 \in I$, the flow φ on $I \times V$ generated by F is the uniquely determined map $\varphi: I \times V$, if it exists, such that

$$\text{(i)} \quad \partial_t \varphi(t,x) = F(\varphi(t,x))$$
$$\text{(ii)} \quad \varphi(0,x) = x.$$

→ For compact $K \subset U$, there exists some interval $[0,c]$, $c > 0$, such that F generates a local flow on $[0,c] \times K$.

Theorem: Let $U \subseteq \mathbb{R}^n$ open and $F: U \to \mathbb{R}^n$ be C^1. Let $V \subseteq U$ open and let $\varphi: I \times V \to U$ be a flow. Then for every t, the map φ_t maps V diffeomorphically onto its image and its derivative $D\varphi_t(x)$ at $x \in V$ is the value $X(t)$ of the solution to the linear matrix IVP

$$\frac{d}{dt} X = A(t) X, \quad A(\tau) = DF(\varphi_\tau(x)),$$
$$X(0) = Id.$$

Def: $g: V \to W$ such that g is bijective, continuously differentiable and g^{-1} is continuously differentiable, is a diffeomorphism.

Proof: Fix $x \in V$, $t > 0$ (wlog). We need to show that

$$\lim_{h \to 0} \frac{(\varphi_t(x+h) - \varphi_t(x) - X(t)h)}{|h|} = 0.$$

Let $y: [0,t] \to U$ be the solution to the ODE $\frac{dy}{d\tau} = F(y)$ with $y(0) = x$ and $y(t) = \varphi_t(x)$. From earlier lemma, we get constants $L, r > 0$.

For $h \in \mathbb{R}^n$ w/ $|h| \leq r e^{-Lt}$, we consider $0 \leq \tau \leq t$,
$$\varphi_\tau(x+h) - \varphi_\tau(x) := \Delta_\tau(h).$$

For $\tau = 0$, $\Delta_0(h) = \varphi_0(x+h) - \varphi_0(x) = x+h-x = h.$

$\Rightarrow$ For $0 \le \tau \le t$, $|\Delta_\tau(h)| = |\varphi_\tau(x+h) - \varphi_\tau(x)| \le |h| e^{L\tau} \le r$.

Since $\tau \mapsto \varphi_\tau(x+h)$, $\tau \mapsto \varphi_\tau(x)$, solutions to the ODE,

$$\frac{d}{d\tau}(\Delta_\tau(h)) = \frac{d}{d\tau}(\varphi_\tau(x+h) - \varphi_\tau(x)) = F(\varphi_\tau(x+h)) - F(\varphi_\tau(x)).$$

Since F is C^1, by Taylor Expansion or FTC, there exists a continuous function $R: [0,t] \times \overline{B(0,r)} \to \mathbb{R}^n$ such that

$R(\tau, 0) = 0$ $\forall$ $0 \le \tau \le t$ and such that $\Delta_\tau(h) \in \overline{B(0,r)}$

$$F(\varphi_\tau(x) + \Delta_\tau(h)) - F(\varphi_\tau(x)) = DF(\varphi_\tau(x))\,\Delta_\tau(h) + |\Delta_\tau(h)|\,R(\tau, \Delta_\tau(h)).$$

Thus, $$\frac{d}{d\tau}(\Delta_\tau(h)) = \underbrace{DF(\varphi_\tau(x))}_{A(\tau)}\Delta_\tau(h) + |\Delta_\tau(h)|\,R(\tau, \Delta_\tau(h)).$$

Denote by $X: [0,t] \to \mathbb{R}^{n\times n}$ the solution to the matrix IVP

$$\begin{cases} \dfrac{dX}{d\tau} = A(\tau)X \\ X(0) = Id \end{cases}$$

Now consider $\gamma(\tau) = \Delta_\tau(h) - X(\tau)h$

$$= (\varphi_\tau(x+h) - \varphi_\tau(x)) - X(\tau)h.$$

Hence $$\frac{d}{d\tau}(\gamma(\tau)) = A(\tau)\Delta_\tau(h) + |\Delta_\tau(h)|\,R(\tau, \Delta_\tau(h)) - A(\tau)X(\tau)h.$$

$$= A(\tau)(\gamma(\tau)) + R(\tau, \Delta_\tau(h))\,|\Delta_\tau(h)|.$$

We have $\gamma(0) = 0$, thus upon integrating in τ

$$\gamma(\tau) = \int_0^\tau A(s)\gamma(s) + R(s, \Delta_s(h))\,\underbrace{|\Delta_s(h)|}_{\le |h| e^{Ls}}\,ds.$$

Now we estimate: let $a = \max_{0\le s\le t} \|A(s)\|_{op}$, $b(r) = \max_{(s,u)\in[0,t]\times \overline{B(0,r)}} |R(s,u)|$.

Then for $0 \le \tau \le t$

$$|\gamma(t)| \leq \int_0^t a\,|\gamma(s)|\,ds + \int_0^t |h| e^{Ls} b(r)\,ds$$

$$\leq a \int_0^t |\gamma(s)|\,ds + |h| e^{Lt} b(r)\, t.$$

Now given $\varepsilon > 0$, we choose $\delta > 0$ so that for $r < \delta$, $b(r) \leq \dfrac{\varepsilon}{t e^{(a+L)t}}$. This is possible because R is continuous and $R(s, 0) = 0$ for all $0 \leq s \leq t$. Then by Gronwall,

$|\gamma(t)| \leq |h| e^{(a+L)t}\, t \cdot b(r)$. Then, finally choose $|h|$ so small such that $|\Delta_s(h)| \leq |h|\, e^{Lt} < \delta$, then we conclude

$$\underbrace{\frac{|\gamma(t)|}{|h|}}_{= \left|\varphi_t(x+h) - \varphi_t(x) - X(t)h\right|} \leq \varepsilon.$$

This gives differentiability.

Now for continuity of $D\varphi_t(\cdot) : V \to \mathbb{R}^{n\times n}$. For $x \in V$, set $x_1 = x$, $x_2 = x + h$, then $D\varphi_t(x_j)$ is the solution to

$$\begin{cases} \dfrac{dX}{dt} = A_j(t)X, & A_j(t) = DF(\varphi_t(x_j)) \\ X(0) = Id \end{cases}$$

$\to$ Gronwall-type argument $\left[\begin{array}{l} \varphi_t \text{ is diffeomorphism onto} \\ \text{its image.} \end{array}\right]$

$\to$ φ_t is injective by uniqueness of IVP, and we've shown that the derivative is continuous. To invoke Inverse Function Theorem, we need to show that $D\varphi_t(x)$ is invertible. Recall

$D\varphi_t(x)$ is the solution to the linear matrix ODE

$$\begin{cases} \dfrac{dX}{dt} = DF(\varphi_t(x)) \cdot X \\ X(0) = Id. \end{cases}$$

Since $X(0)$ is invertible, this property is preserved for all $X(t)$. So, $D\varphi_t(x)$ is invertible. ■

Recall: Liouville's Formula. Let Φ be a fundamental matrix solution for the linear system

$$\frac{dy}{dt} = A(t)y$$

Then, $\frac{d}{dt}\det(\Phi(t)) = tr(A(t))\det(\Phi(t))$.

For a flow, $\varphi: I \times U \to U$, we know that the derivative $D\varphi_t(x)$ satisfies

$$\frac{d}{dt}(D\varphi_t(x)) = \underbrace{DF(\varphi_t(x))}_{A(t)} D\varphi_t(x).$$

Thus by Liouville's,

$$\frac{d}{dt}(\det(D\varphi_t(x))) = \underbrace{Tr(DF(\varphi_t(x)))}_{div(F(\varphi_t(x)))} \det(D\varphi_t(x)).$$

$$= div(F)(\varphi_t(x))$$

$\Rightarrow$ Let say $K \subset U$ is compact, $t \in [0, c]$, then

$$vol(\varphi_t(K)) = \int_{\varphi_t(K)} 1 \, dy = \int_K 1 \cdot |\overset{>0}{\det(D\varphi_t(x))}| \, dx$$

$$= \int_K \det(D\varphi_t(x)) \, dx$$

$$\Rightarrow \frac{d}{dt} vol(\varphi_t(K)) = \underbrace{\int_K (div F)(\varphi_t(x)) \det(D\varphi_t(x)) \, dx}_{0 \text{ when } (div F) \equiv 0}.$$

Divergence free vector fields.

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Stability of Equilibrium points:

Let $F: U \to \mathbb{R}^n$ be a C^1 vector field, U open. Consider

$$\frac{dy}{dt} = F(y).$$

Def: A point $x_0 \in U$ is called an equilibrium solution or a critical point of the ODE if $F(x_0) = 0$.

$\hookrightarrow$ $y(t) = x_0$ is a solution.

Def: A equilibrium point is stable if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x \in B(x_0, \delta)$, the solution $y(t)$ with $y(0) = x$ is defined for all $t > 0$ and $y(t) \in B(x_0, \varepsilon)$ for all t. Otherwise, we call the point unstable.

Def: An equilibrium point $x_0 \in U$ is asymptotically stable if it is stable and if there exists $\delta > 0$ s.t. for all $x \in B(x_0, \delta)$ the solutions converge to the equilibrium point, i.e.

$$\lim_{t \to \infty} y(t) = x_0.$$

Eg: $\frac{dy}{dt} = Ay$; where $A \in \mathbb{R}^{2\times 2}$.

(*) If A has two negative eigenvalues or a conjugate pair of eigenvalues with negative real part, then 0 is asymptotically stable.

(+) If A has complex eigenvalues with 0 real part, then the origin is stable but not asymptotically so.

Linearization: Let $F(x_0) = 0$. Then, by Taylor Expansion

$$F(x) \approx F(x_0) + DF(x_0)(x - x_0).$$

$\nearrow$ How well does this describe $\frac{dy}{dt} = F(y)$ near x_0?

So $\frac{d}{dt}(y(t) - x_0) = \frac{dy}{dt} = F(y) = DF(x_0)(x - x_0) + R.$

1

Theorem: Let $U \subset \mathbb{R}^n$ open and $F: U \to \mathbb{R}^n$ a C^1 vector field. Let $x_0 \in U$ be an equilibrium point for

$$\frac{dy}{dt} = F(y).$$

If every eigenvalue of $DF(x_0) \in \mathbb{R}^{n \times n}$ has negative real part, then the equilibrium point is asymptotically stable.

else, consider $z(t) = y(t) - x_0$

Proof: Let $A = DF(x_0)$, and wlog, let $x_0 = 0$. Let $\mu_0 > 0$ be such that $Re(\lambda) < -\mu_0$ for every eigenvalue λ of A. Choose a constant $C > 0$ so large such that

$\|e^{At}\|_{op} \leq Ce^{-\mu_0 t}$.

Recall: Each entry of e^{At} is a linear combination of $t^k e^{at}\sin(bt)$, $t^k e^{at}\cos(bt)$, where $\lambda = a + ib$, $0 \leq k \leq n-1$.

By Taylor Expansion $F(x) = \overset{DF(0)}{A}x + \overset{\int_0^1 (DF(\eta x) - DF(0))\,d\eta}{R(x)}x$ where $R(x)x \to 0$ as $x \to 0$.

For a solution $y: (\alpha, \beta) \to U$ to the IVP:

$$\frac{dy}{dt} = F(y) = Ay + R(y)y$$

$$y(0) = y_0$$

We compute: $\frac{d}{dt}\left(e^{-At}y(t)\right) = \frac{d}{dt}\left(e^{-At}e^{At}y_0\right)$ So,

$$= -Ae^{-At}y(t) + e^{-At}\underbrace{\frac{dy}{dt}}_{Ay + R(y)y}$$

$$= e^{-At}R(y)y.$$

Now integrate in time from 0 to $t \in [0, \beta)$.

$e^{-At}y(t) - y_0 = \int_0^t e^{-As}R(y)y\,ds$. Now apply e^{At}

$$y(t) = e^{At}y_0 + \int_0^t e^{A(t-s)}R(y(s))y(s)\,ds$$

Thus: $|y(t)| \leq |e^{At}y_0| + \int_0^t |e^{A(t-s)}R(y(s))y(s)|\,ds$

$$\le C e^{-\mu_0 t}|y_0| + C\int_0^t e^{-\mu_0(t-s)} \|R(y(s))\| \cdot |y(s)|\, ds$$

$$\Rightarrow \quad e^{\mu_0 t} y(t) \le C|y_0| + C\int_0^t e^{\mu_0 s}\|R(y(s))\|\cdot|y(s)|\,ds.$$

Let $\overline{B(0,r)}$ be a closed ball around the origin of radius $r>0$ (to be fixed later). Denote $M := \max_{x\in\overline{B(0,r)}} \|R(x)\|_{op}$.

If $y(s)\in\overline{B(0,r)}$ for all $s\in(0,t)$, then by Gronwall,

$$e^{\mu_0 t}|y(t)| \le C|y_0| e^{CMt}.$$

$\Rightarrow \quad |y(t)| \le C\cdot|y_0| e^{(CM-\mu_0)t}$. We choose $r>0$ so small that $CM<\mu_0$, which is possible because $R(x)\to 0$ as $x\to 0$.
↳ and is cont.

Claim: For a maximal integral curve, $y:(\alpha,\beta)\to U$ with $y(0)\in B(0,\delta)$ we have for all $0\le t<\beta$, ~~we have for all~~ $y(t)\in B(0,r)$. Suppose not. Then there is some $0<t^*<\beta$ such that $|y(t^*)|=r$. Up to time t^*, we have $|y(t^*)| \le C\underbrace{|y_0|}_{\le \frac{r}{2C}} \underbrace{e^{(CM-\mu_0)t^*}}_{<1} < \frac{r}{2}$.

$\delta = \frac{r}{2C}$

This contradicts $|y(t^*)|=r$.

Since $|y(t)|\le r$ for all $t\in[0,\beta)$, we have $\beta=\infty$, and the bound implies $y(t)\to 0$.

Def: A Lyapunov function for a critical x_0 of a vector field $F: U\to\mathbb{R}^n$ is a C^1 function $L:U\to\mathbb{R}$ such that:

(i) L has an isolated minimum at x_0, $L(x_0)=0$,
ie, $L(x)>0$ for x in a nbhd of x_0

(ii) $DL(x)F(x)\ge 0$ $\forall x\in U$ or $DL(x)F(x)\le 0$ for all $x\in U$.

Note: If $y(t)$ is a solution to $\frac{dy}{dt}=F(y)$, then

$$\frac{d}{dt}\big(L(y(t))\big) = DL(y(t))\frac{dy}{dt} = DL(y)F(y) \ge 0 \text{ or } \le 0$$

ie increasing or decreasing along $y(t)$.

Theorem: Let $L: U \to \mathbb{R}$ be a Liapunov functional for the critical point x_0 of $\frac{dy}{dt} = F(y)$ with $F: U \to \mathbb{R}^n$ in C^1.
Then we have

a) If $DL(x)F(x) \leq 0$ $\forall x \in U$, Then x_0 is stable

b.) If $DL(x)F(x) < 0$ $\forall x \in U^*$ Then x_0 is asymptotically stable

c.) If $DL(x)F(x) > 0$ $\forall x \in U \setminus \{x_0\}$, then x_0 is unstable.

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Theorem: Let $L: U \to \mathbb{R}$ be a Liapunov function for the critical point x_0 of a locally Lipschitz continuous vector field $F: U \to \mathbb{R}^n$. Then

(a) if $DL(x)F(x) \le 0$ $\forall x \in U$, then x_0 is stable.

(b) if $DL(x)F(x) < 0$ $\forall x \in U\setminus\{x_0\}$, then x_0 is asymptotically stable.

(c) if $DL(x)F(x) > 0$ $\forall x \in U\setminus\{x_0\}$, then x_0 is unstable.

Proof: Let $x_0 \in K \subset U$ be a compact neighborhood of x_0 so that x_0 is the only 0 of the Liapunov functional. Denote the minimum of L on ∂K by $\mu = \min_{x\in\partial K} L(x) > 0$. Let $V = \{x \in K : L(x) < \mu\}$. Then, $x_0 \in V$, so $V \ne \emptyset$, and so is a nonempty neighborhood of x_0 contained in K.

(a) We now show that for any integral curve $y(t)$, $y: [0,\beta) \to U$, with $y(0) \in V$, we have $y(t) \in V$ for all $t \in [0,\beta)$. That is, x_0 is stable. Suppose not. Then there exists $t^* \in [0,\beta)$ such that $y(t) \in V$ for $t < t^*$ but $y(t^*) \notin V$. Since K is closed, we have that $y(t^*) \in K$. By hypothesis, $t \mapsto L(y(t))$ is monotone decreasing, thus $L(y(t^*)) \le L(y(0)) < \mu$. But $L(y(t^*)) \ge \mu$, since $y(t^*) \in V^c$. Thus a contradiction. Furthermore, since $y(t) \in K$ for $t \in [0,\beta)$, we must have $\beta = \infty$. Thus x_0 is stable.

(b) Let $y(t)$ again be an integral curve with $y(0) \in V$. By (a), $y(t): [0,\infty) \to V$ holds for the curve. Now we show $\lim_{t\to\infty} y(t) = x_0$. Since $L(y(t))$ is monotone decreasing and always ≥ 0, $t \mapsto L(y(t))$ has a limit $a \ge 0$ as $t \to \infty$. Define $V_a = \{x \in K : L(x) \ge a\}$. Then $y(t) \in V_a$ for all $t \ge 0$. Denote by M the maximum of $x \mapsto DL(x)F(x)$ on compact V_a. Then, $\frac{d}{dt}(L(y(t))) = DL(y(t)) \cdot F(y(t)) \le M$ for all $t \ge 0$. Now integrate:

$$L(y(t)) - L(y(0)) \le t \cdot M.$$

If $a > 0$, then $x_0 \notin V_a$ and thus $M < 0$ because $DL(x)F(x) < 0$ for $x \ne x_0$ in a compact neighborhood of x_0. But then L would be unbounded from below on the compact set V_a. Thus $a = 0$, so ~~lim~~ $y(t) = 0$, and x_0 is the only 0 of $L \in K$, so

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$\lim_{t\to\infty} y(t) = x_0$.

c) See in lecture notes.

5 (The Liénard equation). Consider

$$z'' + f(z)z' + z = 0, \quad z: \mathbb{R}\to\mathbb{R} \quad \text{w/ } f \le 0 \text{ or } f \ge 0.$$

assume $f: \mathbb{R}\to\mathbb{R}$ is $C^1(\mathbb{R})$. Set $x = z$, $y := z'$. Then

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -f(x)y + x \end{pmatrix} = F(x,y), \quad F(0,0) = 0.$$

Try: $L(x,y) = x^2 + y^2$

Then: $DL(x,y)F(x,y) = [2x, 2y]\begin{bmatrix} y \\ -f(x)y - x \end{bmatrix}$

$$= 2xy + 2y(-f(x)y - x)$$

$$= -2y^2 f(x).$$

($y=0$ someone? $DF(x)L(y) \ge 0$)

Thus if $f(x) > 0$ stable, $\overbrace{f(x) > 0,\ 0\text{-stable}}^{\text{not quite}}$ if $f(x) \le 0$, so unstable.

<u>On the Stable Manifold Theorem:</u>

$$\frac{dy}{dt} = F(y); \quad F(0) = 0, \to \text{Taylor } \frac{dy}{dt} = DF(0)y + G(y).$$

PDE:

Informal definition: A partial differential equation is an equation of an unknown function of several variables and some of its partial derivatives.

Eg: Fundamental Linear Equations:

- Transport $u:[0,\infty)\times\mathbb{R}^n \to \mathbb{R}$, $b\in\mathbb{R}^n$

$$\partial_t u + \sum_{j=1}^{n} b_j \partial_j u = 0$$

- Laplace equation: $u:\mathbb{R}^n\to\mathbb{R}$ (or $\mathbb{C}$)

$$\underbrace{\Delta u}_{=\sum_{j=1}^{n}\partial_j^2 u} = 0$$

- Nonhomogeneous version

$$-\Delta u = f$$

- Heat equation: $u:[0,\infty)\times\mathbb{R}^n\to\mathbb{R}$ (or $\mathbb{C}$)

$$\partial_t u - \Delta u = 0.$$

- Wave: For $u:[0,\infty)\times\mathbb{R}^n\to\mathbb{R}$ (or $\mathbb{C}$)

$$-\partial_t^2 u + \Delta u = 0.$$

- Schrodinger Equation $u:[0,\infty)\times\mathbb{R}^n\to\mathbb{C}$, given $V:\mathbb{R}^n\to\mathbb{R}$,

$$i\partial_t u = (-\Delta + V)u.$$

Eg: Nonlinear PDE:

- Minimal surface: $u:\mathbb{R}^n\to\mathbb{R}$,

$$\sum_{j=1}^{n}\partial_j\left(\frac{\partial_j u}{(1+\sum_{j=1}^{n}(\partial_j u)^2)^{1/2}}\right) = 0.$$

$$\leftrightarrow \nabla\cdot\left(\nabla u\right.$$

• Korteweg-de Vries (KdV) equation For $u: [0,\infty) \times \mathbb{R} \to \mathbb{R}$

$$\partial_t u + \partial_x^3 u - 6u\partial_x u = 0$$

<u>Basic Concepts</u>: When we solve a PDE, we usually look for a solution that satisfies certain prescribed conditions (often called data). Important settings:

→ Boundary value problems (BVP) For an equation such as Laplace or Poisson on a bounded domain $U \subseteq \mathbb{R}^n$, often, data are typically prescribed on the boundary:

- Dirichlet $-\Delta u = f$ in U, $u = g$ on ∂U
- Neumann: $-\Delta u = f$ in U, $\eta \cdot \nabla u = g$ on ∂U, where η = outward unit normal to ∂U

→ Initial Value Problem (Cauchy) For evolution equations, the problem is usually to start with given data at $t=0$ and to find a solution that agrees with data at $t=0$.

<u>Wave</u> $-\partial_t^2 u + \Delta u = 0$, $u(0,x) = g(x)$, $\partial_t u(0,x) = h(x)$

We say a BVP or IVP is <u>well-posed</u> if:

- Existence: For any given data, there exists a solution
- Uniqueness: For any given data, there is at most one solution
- Continuous dependence on data.

<u>Terminology</u> • Multi-index: $\alpha \in (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$;
$|\alpha| = \alpha_1 + \ldots + \alpha_n$

$$\partial^\alpha u = D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

• The <u>order</u> of a PDE is the highest derivative that occurs.

• Classification

- Linear: $\sum_{|\alpha| \le k} a_\alpha(x) \partial^\alpha u(x) = 0$

- Semi-linear: This is when the PDE is linear in the highest order derivative with coefficients that do not depend on u. i.e.,

$$\sum_{|\alpha| = k} a_\alpha(x)\, \partial^\alpha u(x) + F(D^{k-1}u, \ldots, u, x) = 0$$

- Quasi-linear: When the PDE is still linear in the highest order derivative w/ coefficients that depend on at most $k-1$ derivatives.

$$\sum_{|\alpha| = k} a_\alpha(D^{k-1}u, \ldots, Du, u, x)\, \partial^\alpha u(x) + \ldots + F(D^{k-1}u, \ldots, u, x) = 0.$$

- Else, fully nonlinear.

Interlude: Action-Principle (Classical)

→ Many PDEs arise from the "Action Principle".

Laplace: Let $U \in \mathbb{R}^n$ open and bounded. Consider the Dirichlet energy functional for $u: U \to \mathbb{R}$:

$$S[u] = \frac{1}{2}\int_U |\nabla u|^2 dx.$$

To minimize, look at "small" smooth, compactly supported perturbation of u: i.e.; we require

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} S[u+\varepsilon\varphi] = 0.$$

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} S[u+\varepsilon\varphi] = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int \frac{1}{2} |\nabla u + \varepsilon \nabla\varphi|^2 dx$$

$$= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \int \frac{1}{2}\left(|\nabla u|^2 + 2\varepsilon \nabla u \cdot \nabla\varphi + \varepsilon^2 |\nabla\varphi|^2\right) dx$$

$$= \int \nabla u \cdot \nabla \varphi\, dx = \int_U (-\Delta u)\varphi\, dx \overset{!}{=} 0 \Rightarrow \boxed{-\Delta u = 0}\ (\text{a.e.})$$

→ Euler-Lagrange for $S[u]$.

Summary of Variational Calculus.

③

Transport Equation:

Motivation: Physics: Denote by $v : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ given velocity vector field. Denote $\rho : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ is the density of a gas at position x at time $t \geq 0$. We assume gas moves at velocity $v(t,x)$ at position $x \in \mathbb{R}^n$, $t \geq 0$.

Conservation of mass:
The mass in a fixed volume $B \subseteq \mathbb{R}^n$ can only change by inward or outward flow through the boundary.

$$\frac{\partial}{\partial t} \int_B \rho(t,x)\,dx = -\int_{\partial B} \eta(x) \cdot v(t,x) \cdot \rho(t,x)\,dS(x)$$

$$= -\int_B \nabla \cdot (v(t,x)\rho(t,x))\,dx$$

Thus: $\int_B \frac{\partial}{\partial t}\rho(t,x) + \nabla \cdot (v(t,x)\rho(t,x))\,dx = 0.$

Since B is arbitrary, we arrive at

$$\frac{\partial}{\partial t}\rho(t,x) + \nabla \cdot (v(t,x)\rho(t,x)) = 0.$$

For fixed $v(t,x) = b \in \mathbb{R}^n$, we consider IVP:

$$\begin{cases} \partial_t u + b \cdot \nabla u = 0 & \text{on } (0,\infty) \times \mathbb{R}^n \\ u(0,x) = g(x) & g \in C^1(\mathbb{R}^n) \end{cases}$$

Heuristically: Observe $\partial_t + b \cdot \nabla = \binom{1}{b} \cdot \binom{\partial_t}{\nabla}$ is a directional derivative in direction $\binom{1}{b}$, so solutions are constant along the direction vector, thus we can fill the whole space.

Set $z(s) := u(t+s, x+bs)$, $b \in \mathbb{R}^n$, $\frac{dz}{ds} = \partial_t u + b \cdot \nabla u(t+s, x+bs)$
$= (\partial_t u + b \cdot \nabla u)(t+s, x+bs)$
$= 0.$

Thus, $z(s)$ is constant $\Rightarrow$
$z(s) = z(-t) = u(0, x-bt) = g(x-bt)$. $\Rightarrow u(t,x) = z(0) = g(x-bt)$

Nonhomogeneous version:

$$\begin{cases} \partial_t u + b\cdot\nabla u = f & , f \in C^1(\mathbb{R}^n) \\ u(0,x) = g(x) & g \in C^1(\mathbb{R}^n) \end{cases}$$

Again, let $z(s) := u(t+s, x+bs)$. Then $\frac{d}{ds}z(s) = (\partial_t u + b\cdot\nabla u)(t+s, x+bs) = f(t+s, b+s)$.

$$u(t,x) = z(0) = z(-t) + \int_{-t}^{0} \frac{d}{ds} z(s)\,ds$$

$$= g(x-bt) + \int_{-t}^{0} f(t+s, x+bs)\,ds. \text{ Thus:}$$

$$u(t,x) = g(x-bt) + \int_{-t}^{0} f(t+s, x+bs)\,ds; \quad \tau = t+s$$

$$u(t,x) = g(x-bt) + \int_0^t f(\tau + x(\tau - t))\,d\tau \Rightarrow \text{Final:}$$

$$\boxed{u(t,x) = g(x-bt) + \int_0^t f(\tau, x+b(\tau-t))\,d\tau}$$

Proposition: Let $f \in C^1((0,\infty), \mathbb{R}^n) \cap C([0,\infty), \times \mathbb{R}^n)$, $g \in C^1(\mathbb{R}^n)$. Then for any fixed vector $b \in \mathbb{R}^n$, the IVP

$$\begin{cases} \partial_t u + b\cdot\nabla u = f & \text{in } (0,\infty)\times\mathbb{R}^n \\ u(0,x) = g(x) \end{cases}$$

has the unique solution:

$$u(t,x) = g(x-tb) + \int_0^t f(s, x+b(s-t))\,ds.$$

with $u \in C^1([0,\infty)\times\mathbb{R}^n) \cap C([0,\infty), \mathbb{R}^n)$.

Proof: u is a solution to IVP problem by direct computation, and u satisfies regularity.

Uniqueness: If v_1, v_2 are solutions, then:

$$v := v_1 - v_2 \text{ satisfy } \partial_t + b\nabla(v) = 0, \; v(0,x) = 0 \text{ on } \mathbb{R}^n$$

so $v \equiv 0$ by previous computations.

Laplace and Poisson:

Preparation: Mollifiers are a tool to build smooth approximations of given functions.

Def: Let $\eta \in C^\infty(\mathbb{R}^n)$, $\eta \geq 0$, $supp(\eta) \subseteq B(0,1)$ and satisfies

$$\|\eta\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \eta(x)\,dx = 1.$$

Then, for $\varepsilon > 0$, we define the mollifier $\eta_\varepsilon(x) = \varepsilon^{-n}\eta(\frac{x}{\varepsilon})$ $x \in \mathbb{R}^n$.

so $\eta_\varepsilon \in C^\infty(\mathbb{R}^n)$, $supp(\eta_\varepsilon) \subset B(0,\varepsilon)$ and

$$\int_{\mathbb{R}^n} \eta_\varepsilon(x)\,dx = \int_{\mathbb{R}^n} \varepsilon^{-n}\eta(\tfrac{x}{\varepsilon})\,dx = 1$$

Eg. $\eta(x) = \begin{cases} C\cdot\exp(-\frac{1}{1-|x|^2}) & |x|<1 \\ 0 & |x|\geq 1 \end{cases}$, C chosen so that $\int \eta(x) = 1$.

Theorem (Properties of mollifiers)

Let $U \subseteq \mathbb{R}^n$ open, $\varepsilon > 0$, $U_\varepsilon := \{x \in U : dist(x, \partial U) > \varepsilon\}$, fix η_ε $\varepsilon > 0$.

For $f \in L^1_{loc}$, $x \in U_\varepsilon$, we define the mollification of f by

change of variables

$$f_\varepsilon = \int_{\mathbb{R}^n} \eta_\varepsilon(x-y) f(y)\,dy = \int_{\mathbb{R}^n} \eta_\varepsilon(y) f(x-y)\,dy$$

$$= \int_{B(0,\varepsilon)} \eta_\varepsilon(y) f(x-y)\,dy$$

Then (i) f_ε is a smooth function of $C^\infty(U_\varepsilon)$

(ii) $f_\varepsilon \to f$ a.e. as $\varepsilon \to 0$.

(iii) If $f \in C(U)$, then $f_\varepsilon \to f$ uniformly on compact subsets of U

(iv) If $1 \leq p < \infty$, $f \in L^p_{loc}(U)$, then $f^\varepsilon \to f$ in $L^p_{loc}(U)$.

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Convolution g, f. $(g*f)(x) = \int_{\mathbb{R}^n} g(y) f(x-y)\,dy$

$= \int_{\mathbb{R}^n} g(x-y) f(y)\,dy = (f*g)(x)$.

Proof:

(i) Fix $x \in U_\varepsilon$, $i \in \{1, \dots, n\}$, let $h \in \mathbb{R}$ s.t. $x+he_i \in U_\varepsilon$. Then

$$\frac{1}{h}\left(f^\varepsilon(x+he_i) - f^\varepsilon(x)\right) = \frac{1}{\varepsilon^n}\frac{1}{h}\int_U \left(\eta_\varepsilon\left(\frac{x+he_i-y}{\varepsilon}\right) - \eta\left(\frac{x-y}{\varepsilon}\right)\right) f(y)\,dy.$$

$B(x+he_i,\varepsilon)$ $\quad B(x,\varepsilon)$

Consider $h_n := \frac{1}{n}$.

→ We can restrict integration over some compact subset of U, $K \subseteq U$, w/ $B(x,\varepsilon), B(x+he_i,\varepsilon) \subseteq K$ uniformly for all small h.
To apply DCT: want to have

MVT if necessary

$$\frac{1}{h}\left(\eta_\varepsilon\left(\frac{x+he_i-y}{\varepsilon}\right) - \eta\left(\frac{x-y}{\varepsilon}\right)\right) \to \frac{1}{\varepsilon}\frac{\partial\eta}{\partial x_i}\left(\frac{x-y}{\varepsilon}\right)$$

converges uniformly on

is uniformly bounded for $y \in K$ for all suff. small h.

Thus, by DCT, we obtain

$\frac{\partial f^\varepsilon}{\partial x_i}(x)$ exists and equals

$$\frac{\partial f^\varepsilon}{\partial x_i}(x) = \int_U \frac{\partial \eta_\varepsilon}{\partial x_i}(x-y) f(y)\,dy = \left(\frac{\partial \eta_\varepsilon}{\partial x_i} * f\right)(x)$$

By iteration, $D^\alpha f^\varepsilon\big|_{U_\varepsilon} = (D^\alpha \eta_\varepsilon * f)\big|_{U_\varepsilon}$.

(ii) Here, we use Lebesgue differentiation theorem.

For $f \in L^1_{loc}$, we have a.e. $x \in \mathbb{R}^n$, $\lim_{r \to 0} \frac{1}{m(B(r,x))}\int_{B(x,r)} |f(y) - f(x)|\,dy = 0$.

shrink $\quad \int_{B(x,r)} |f(y)-f(x)|\,dy \to 0$

①

For all Lebesgue points in U, $\quad f(x) = \varepsilon^{-n}\int_{B(x,\varepsilon)} \eta\left(\frac{x-y}{\varepsilon}\right) f(x)\,dy \quad \|\eta_\varepsilon\| = 1.$

$$|f_\varepsilon(x) - f(x)| = \left|\int_{B(x,\varepsilon)} \varepsilon^{-n}\eta\left(\frac{x-y}{\varepsilon}\right)(f(y)-f(x))\,dy\right|.$$

$$\le \varepsilon^{-n}\int_{B(x,\varepsilon)} \underbrace{\left|\eta\left(\frac{x-y}{\varepsilon}\right)\right|}_{\le \|\eta\|_\infty}\cdot|f(y)-f(x)|\,dy$$

$$= \frac{C_n}{|B(x,\varepsilon)|}$$

$$\le C_n \|\eta\|_{L^\infty} ⨍_{B(x,\varepsilon)} |f(y)-f(x)|\,dy \longrightarrow 0 \text{ by } x \text{ is a Lebesgue point}$$

$$m(B(x,\varepsilon)) = \alpha(n)\,\varepsilon^n.$$ ■

(iii), (iv) = HW reading.

Physical origins of Laplace Equation:

Often comes up to describe that some quantity, $u(x)$ modeling the density of a substance is in equilibrium: (Temp. chem concentration).

For any smooth region $B \subseteq \mathbb{R}^n$, in equilibrium, the net flux of u through the boundary is 0:

$$\int_{\partial B} F\cdot\nu\,dS(x), \quad F \text{ is flux density of } u.$$

$$0 \underset{\text{by assumption}}{=} \int_{\partial B} F\cdot\nu\,dS(x) = \int_B \nabla\cdot F\,dx, \text{ since } B \text{ is arbitrary}$$

$$\operatorname{div}(F) = 0 \qquad \nabla u = \text{highest ascent.}$$

Heuristically in many situations, a reasonable approx is that $F = -a\nabla u$, $a>0$. Thus
$$0 = \nabla\cdot F = -a\Delta u \Rightarrow \Delta u = 0$$ ■

Poissons: $-\Delta u = f$.

Properties of harmonic functions:

Def: Let $U \subset \mathbb{R}^n$, $u \in C^2(U)$ we say u is harmonic on U if $\Delta u = 0$.

Eg: $n=1 \Rightarrow \frac{\partial^2}{\partial x^2} u = 0 \Rightarrow u(x) = ax+b$.

(ii) arbitrary n.

$$u(x_1, \ldots, x_n) = c_0 + \sum_{j=1}^{n} c_j x_j + \sum_{i \neq j} c_{ij} x_i x_j$$

(iii) $n=2$ $\mathbb{R}^2 \cong \mathbb{C}$: recall, for a holomorphic function,

$f: \mathbb{C} \to \mathbb{C}$, $f(x_1 + ix_2)$. $Re(f) = u(x_1, x_2)$
$Im(f) = v(x_1, x_2)$

Satisfy Cauchy Riemann equations:

$$\frac{\partial u}{\partial x_1} = \frac{\partial v}{\partial x_2}\ ; \quad \frac{\partial u}{\partial x_2} = -\frac{\partial v}{\partial x_1}$$

$$\Rightarrow \Delta u = \partial_1^2 u + \partial_2^2 v = \partial_1 \partial_2 v - \partial_2 \partial_1 v = 0.$$

$\Rightarrow \Delta v = 0$ as well.

Recall Cauchy Integral Formula: f = holomorphic

$$f(z) = \frac{1}{2\pi i} \int_{\partial B(r,z)} \frac{f(\xi)}{\xi - z} d\xi.$$

Harmonic functions in arbitrary dimensions enjoy many of the nice properties of holomorphic functions of $n=2$.

<u>Theorem</u>: Let $U \subseteq \mathbb{R}^n$, $u \in C^2(\mathbb{R}^n)$. Then, u is harmonic of U iff $u(x) = \fint_{\partial B(x,r)} u(y)\,dy. =$

$$\frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y)\,dS(y) \quad \text{for all } x \in U,\ r > 0 \text{ s.t. } B(x,r) \subset\subset U.$$

In either case: $\fint_{\partial B(x,r)} u(y)\,dS(y) = \fint_{B(x,r)} u(y)\,dy$

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Theorem: Let $\mathcal{U} \in \mathbb{R}^n$ open and $u \in C^2(\mathcal{U})$. Then u is harmonic iff

$$u(x) = \fint_{\partial B(x,r)} u(y)\,dS(y) = \frac{1}{m(\partial B(x,r))} \int_{\partial B} u(y)\,dS(y)$$

for all x and all $r>0$ s.t. $\overline{B(x,r)} \subseteq \mathcal{U}$. We also have

$$\fint_{\partial B(x,r)} u(y)\,dS(y) = \fint_{B(x,r)} u(y)\,dy$$

Recall $|B(0,r)| = \alpha(n) r^n$, $|\partial B(0,r)| = n\alpha(n) r^{n-1} = \frac{n}{r}|B(0,r)|$, $\alpha(n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$.

Lemma: Let $\mathcal{U} \subseteq \mathbb{R}^n$, $x \in \mathcal{U}$, $R>0$ s.t. $B(x,R) \subset \mathcal{U}$. For $0<r<R$, $u \in C^2(\mathcal{U})$,
set

$$\varphi(r,x) := \fint_{\partial B(x,r)} u(y)\,dS(y). \quad \text{Then,}$$

$$\frac{\partial \varphi}{\partial r}(r,x) = \frac{r}{n} \fint_{B(x,r)} (\Delta u)(y)\,dy, \quad \lim_{r\to 0} \varphi(r,x) = u(x).$$

$\to dS(y) = r^{n-1} dS(z)$

Proof: Let $y = x + rz \Rightarrow \varphi(r,x) = \fint_{\partial B(0,1)} u(x+rz)\,dS(z)$. Thus:

$$\frac{\partial \varphi}{\partial r}(r,x) = \fint_{\partial B(0,1)} (\nabla u)(x+rz)\cdot z\,dS(z)$$

$$= \fint_{\partial B(x,r)} (\nabla u)(y) \cdot \underbrace{\left(\frac{y-x}{r}\right)}_{\nu(y)} dS(y)$$

div Thm $\to$ $$= \frac{\partial \varphi}{\partial r}(r,x) = \frac{1}{|\partial B(0,r)|} \int_{B(x,r)} \overbrace{\operatorname{div}(\nabla u(y))}^{\Delta u}\,dy =$$

$$= \frac{1}{|B(x,r)|} \cdot \frac{r}{n} \int_{B(x,r)} (\Delta u)(y)\,dy \quad \checkmark.$$

Moreover:

$$|\varphi(r,x)-u(x)| \leq \fint_{\partial B(0,1)} \underbrace{|u(x+rz)-u(x)|}_{\leq 2\cdot\|u\|_{L^\infty(\overline{B(x,1)})}}\,dS(z) \longrightarrow 0$$

$\to$ dominated convergence.

■

Proof of Theorem: ($\Rightarrow$) assume u is harmonic. Then for fixed $x \in U$, small $r>0$,

$$\frac{\partial \varphi}{\partial r}(r,x) = \frac{r}{n}\fint_{B(x,r)} \overbrace{(\Delta u)(y)}^{0}\,dy = 0$$

Thus, $\varphi(r,x)$ is constant w.r.t. r thus,

$$\varphi(r,x) = \lim_{r\to 0}\varphi(r,x) = u(x). \text{ (see def of } \varphi\text{)}.$$

($\Leftarrow$) Suppose u has the mean value property. For fixed x, $r>0$ (small),

$$u(x) = \fint_{\partial B(r,x)} u(y)\,dS(y) =: \varphi(r,x).$$

ie, $r \longmapsto \varphi(r,x)$ is constant and equal to $u(x)$. Then,

$$0 = \frac{\partial \varphi}{\partial r}(r,x) = \frac{r}{n}\fint_{B(x,r)} (\Delta u)(y)\,dy.$$ If $\Delta u \not\equiv 0$ in U, then $\exists\, x_0 \in U$

s.t. $(\Delta u)(x_0) > 0$ (wlog), then since $u \in C^2(U)$, Δu is continuous. Then we can find a ball $B(x_0,\tilde{r})$ s.t. $\Delta u|_{B(x_0,\tilde{r})} > 0$. But this gives a contradiction.

Finally, note

$$\fint_{B(x,r)} u(y)\,dy \overset{\text{using polar}}{=} \frac{1}{|B(x,r)|}\int_0^r \underbrace{\left(\int_{\partial B(x,s)} u(y)\,dS(y)\right)}_{|\partial B(x,s)|\,u(x)} ds$$

$$= u(x)\,\frac{1}{|B(x,r)|}\int_0^r |\partial B(x,s)|\,ds = u(x)\cdot\frac{1}{|B(x,r)|}\int_0^r n\alpha(n)\,s^{n-1}\,ds$$

$$= u(x)\frac{1}{|B(x,r)|}\underbrace{\alpha(n)r^n}_{|B(x,r)|} = u(x) = \fint_{\partial B(x,r)} u(y)\,dS(y).$$

MVP.

Theorem (Maximum Principle) $U \subseteq \mathbb{R}^n$ be open and bounded. Suppose $u \in C^2(U) \cap C(\bar{U})$ and is harmonic in U. Then,

(i) (weak max principle) $\max_{\bar{U}} u = \max_{\partial U} u$.

(ii) (Strong max principle) If there exists $x_0 \in U$ s.t.

$u(x_0) = \max_{x \in \bar{U}} u(x)$, then u is constant on the connected component of U containing x_0.

Proof: Suppose there exists $x_0 \in U$ w/ $u(x_0) = \max_{x \in \bar{U}} u(x) =: M < \infty$.

Then for any small $r > 0$, MVP gives

$M = u(x_0) = \fint_{B(x_0,r)} \underbrace{u(y)}_{\leq M}\,dy \leq M$. Since u is cont, we have that

$u(y) = M \quad \forall y \in B(x_0,r)$. Thus:

$A := \{x \in U : u(x) = M\}$ is open, and relatively closed by continuity of u. Thus, A is the connected component of U containing x_0. ▨

Remark: Since $-u$ is also harmonic, applying the max principle to $-u$, we obtain an analogous minimum principle for u. ($\min_{x \in \bar{U}} u = \min_{x \in \partial U} u$)

Corollary: Uniqueness for Dirichlet problem.

Let $U \subseteq \mathbb{R}^n$ be open and bounded. Let $g \in C(\partial U)$, $f \in C(U)$. Then there exists at most one solution $u \in C^2(U) \cap C(\bar{U})$ to the BVP:

$$\begin{cases} -\Delta u = f & \text{in } U. \\ u = g & \text{on } \partial U. \end{cases}$$

Proof: Suppose u_1, u_2 are two solutions to the problem. Then the difference $w := u_1 - u_2$

$$\begin{cases} -\Delta w = 0 \\ w = 0 \quad \text{on } \partial U. \end{cases}$$

By max: $\max_{\bar{U}} w = \max_{\partial U} = 0 \Rightarrow w \leq 0$. Likewise,

$\min_{\bar{U}} w = \min_{\partial U} = 0 \Rightarrow w \geq 0 \Rightarrow w \equiv 0$. ∎

Theorem: Let $U \in \mathbb{R}^n$ open, u harmonic on U. Then $u \in C^\infty(U)$.

Proof: Let η be a mollifier that is radially symmetric (i.e. constant on spheres). Set:

$$u_\varepsilon := \eta_\varepsilon * u \quad \text{in } U_\varepsilon := \{x \in U : \operatorname{dist}(x, \partial U) > \varepsilon\}. \text{ Claim:}$$

$u_\varepsilon = u$. Indeed: for $x \in U_\varepsilon$,

$$u_\varepsilon(x) = \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)\, u(y)\, dy = \frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} \eta\left(\frac{x-y}{\varepsilon}\right) u(y)\, dy$$

$$= \frac{1}{\varepsilon^n} \int_0^\varepsilon \int_{\partial B(x,r)} \underbrace{\eta\left(\frac{x-y}{\varepsilon}\right)}_{\tilde{\eta}\left(\frac{r}{\varepsilon}\right)} u(y)\, dy\, dr = \frac{1}{\varepsilon^n} \int_0^\varepsilon \tilde{\eta}\left(\frac{r}{\varepsilon}\right) \underbrace{\int_{\partial B(x,r)} u(x)\, dy}_{= u(x) \cdot |\partial B(x,r)|}$$

$$= u(x) \cdot \frac{1}{\varepsilon^n} \int_0^\varepsilon \underbrace{\tilde{\eta}\left(\frac{r}{\varepsilon}\right)}_{\eta\left(\frac{x-y}{\varepsilon}\right)} |\partial B(x,r)|\, dr = u(x) \cdot \underbrace{\frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} \eta\left(\frac{x-y}{\varepsilon}\right) dy}_{1}$$

$$= u(x).$$

Thus: $u_\varepsilon \equiv u$. ∎

Theorem: Every bounded harmonic function on $\mathbb{R}^n$ is constant.

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Proof: Let $u: \mathbb{R}^n \to \mathbb{R}$ be bounded and harmonic, i.e., $u \in C^\infty(\mathbb{R}^n)$.

$\Rightarrow \Delta u = 0 \Rightarrow \Delta(\partial_i u) = 0$. Thus, $\partial_i u$ is harmonic. So by MVP for all $x \in \mathbb{R}^n$, all $r > 0$,

$$\partial_i u(x) = ⨍_{B(x,r)} \partial_i u(y)\,dy$$

$$= \frac{1}{|B(x,r)|} \int_{B(x,r)} \underbrace{\partial_i u(y)}_{\operatorname{div}\begin{pmatrix} 0 \\ \vdots \\ u \\ 0 \end{pmatrix} \leftarrow i^{th} \text{ part}}\,dy$$

$$= \frac{1}{|B(x,r)|} \int_{\partial B(x,r)} u(y)\, \nu_i(y)\, ds(y)$$

$$= \frac{1}{\frac{r}{n}|\partial B(x,r)|} \int_{\partial B(x,r)} u(y)\, \nu_i(y)\, ds(y)$$

$$\Rightarrow |\partial_i u(x)| \le \frac{n}{r} ⨍_{\partial B(x,r)} \underbrace{|u(y)|}_{\le \|u\|_\infty} \cdot 1\, ds(y) \le \frac{n}{r} \cdot \|u\|_\infty. \text{ Holds for all } r,$$

so $|\partial_i u(x)| = 0$. Thus, for $i = 1, \dots, n$, $\partial_i u \equiv 0$ on $\mathbb{R}^n$.

Thus, u must be constant. ∎

Theorem: (Harnack's Inequality). Let $U \subseteq \mathbb{R}^n$ be open. For all $V \subset\subset U$ ($\overline{V} \subset U$, $\overline{V}$ = compact) open and connected, there exists a constant $C = C(V) \ge 1$ such that for all non-negative harmonic functions $u \in C^2(U)$

$$\sup_V u \le C \cdot \inf_V u.$$

Hence, $\forall x, y \in V$, $\frac{1}{C} u(y) \le u(x) \le C u(y)$.

Proof: Let $r := \frac{1}{4} \operatorname{dist}(V, \partial U)$. Choose $x, y \in V$ w/ $|x-y| \le r$. Then,

$$u(x) = ⨍_{B(x,2r)} u(z)\,dz \ge \frac{1}{|B(x,2r)|} \int_{B(y,r)} u(z)\,dz = \underbrace{\frac{|B(y,r)|}{|B(x,2r)|}}_{\frac{\alpha(n) r^n}{\alpha(n)(2r)^n}} \cdot \underbrace{\frac{1}{|B(y,r)|} \int_{B(y,r)} u(z)\,dz}_{u(y)} = \frac{1}{2^n} u(y).$$

Thus, for $x, y \in V$ w/ $|x-y| \le r$,

$$2^n u(y) \ge u(x) \ge \frac{u(y)}{2^n}.$$

Now $\bar{V}$ is compact and path connected. So we can cover $\bar{V}$ by a (finite chain) of balls, $B(y_1, \frac{r}{2}), \ldots, B(y_N, \frac{r}{2})$, and such that $B_i \cap B_{i+1} \neq \emptyset$. For all $x, x' \in V$, iteration of the above analysis:

$$u(x) \ge 2^{-Nn} u(x').$$

The theorem follows by $\inf_{x \in V}$ and $\sup_{x \in V}$.

Poisson's Equation: $(-\Delta u = f)$

Goal: Determine explicit representation formula for a solution to Poisson equation for a suitable class of functions f and deduce consequences.

→ δ_0 is a distribution, i.e. $\delta_0 : C_c^\infty(\mathbb{R}^n) \to \mathbb{R}$, $f \mapsto f(0)$

Dirac delta.

→ For $g \in L^1_{loc}$, $f \mapsto \int g(x) f(x)\,dx$, is well-defined and linear.

→ Look for $\underbrace{-\Delta \Phi = \delta_0}_{\text{make sense of this.}}$. Then $u(x) = (\Phi * f)(x)$ is such that $-\Delta_x u = f(x)$.

→ Claim: Look for radially symmetric function $\Phi(x) = v(r)$, $r = |x|$ that is harmonic $\mathbb{R}^n \setminus \{0\}$ ⇒

$$\underbrace{v''(r) + \frac{n-1}{r} v'(r) = 0,}_{HW \; = \Delta(v(|x|))} \quad r > 0.$$

$$\Rightarrow \frac{d}{dr}\left(v'(r) \underbrace{\exp\left(\int_1^r \frac{n-1}{s}\,ds\right)}_{\exp((n-1)\log(r)) \,=\, r^{n-1}} \right) = 0 \Rightarrow \frac{d}{dr}\left(v'(r)\, r^{n-1} \right) = 0,$$

so $v'(r) = \frac{K}{r^{n-1}} \Rightarrow$ for $n \ge 2$.

$$v(r) = \begin{cases} b\log(r) + c & n=2 \\ \frac{b}{r^{n-2}} + c & n>2 \end{cases} \quad ; \ b, c \text{ constants.}$$

Def: The function $\Phi(x) = \begin{cases} \frac{1}{-2\pi} \log(|x|), & n=2 \\ \frac{1}{n(n-2)\,\alpha(n)} \frac{1}{|x|^{n-2}}, & n \geq 3 \end{cases}$

for $x \neq 0$ is called the fundamental solution to the Laplacian.

Idea: Prove existence of Poisson's equation on $\mathbb{R}^n$.

Theorem: Let $n \geq 2$, $f \in C_c^2(\mathbb{R}^n)$. Then

(#) $\quad u(x) := (\Phi * f)(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\,dy$

is in $C^2(\mathbb{R}^n)$ and $-\Delta u = f$ on $\mathbb{R}^n$.

For the integral to make sense, we need $\Phi \in L^1_{loc}(\mathbb{R}^n)$:

$$|\Phi(x)| \leq C \begin{cases} \log(|x|) & n=2 \\ \frac{1}{|x|^{n-2}} & n \geq 2 \end{cases}$$

$$\hookrightarrow \int_0^1 \begin{Bmatrix} \log(r) \\ \frac{1}{r^{n-2}} \end{Bmatrix} r^{n-1}\,dr \underset{\substack{\uparrow \\ 2D}}{=} \int_0^1 \{\log(r)\cdot r \text{ or } r\}\,dr < \infty.$$

$\rightarrow \partial_j \Phi(x) = -\frac{1}{n\alpha(n)} \frac{x_j}{|x|^n}$

$\partial_j \partial_k \Phi(x) = \frac{-1}{n\alpha(n)} \left(\frac{\delta_{j,k}}{|x|^n} - n \frac{x_j x_k}{|x|^{n+2}} \right)$. Then

$|\partial_j \Phi(x)| \leq C \frac{1}{|x|^{n-1}}$ — locally integrable

$|\partial_j \partial_k \Phi(x)| \leq C \frac{1}{|x|^n}$ — Not locally integrable.

Thus, $\int_{\mathbb{R}^n} \underbrace{(\Delta\Phi)(x-y)}_{\text{not locally integrable.}} f(y)\, dy$ is not well-defined

The Fourier Transform on $\mathbb{R}^n$

__Motivation__: Let $A: \mathbb{R}^n \to \mathbb{R}^n$ be linear diagonalizable, i.e., assume there is an orthonormal basis of eigenvectors of A, $Av_j = \lambda_j v_j$. Write $w \in \mathbb{R}^n$ as $w = \sum_{1}^{n} \langle v_j, w \rangle v_j$. Then,

$$Aw = \sum_{j=1}^{n} \langle v_j, w \rangle \lambda_j v_j.$$

$\Rightarrow$ In the cases: $\begin{cases} -\Delta u = 0 \\ (\partial_t - \Delta) u = 0 \\ (\partial_t^2 - \Delta) u = 0 \end{cases}$ building blocks are partial derivatives where ∂_{x_j} acts as multiplication.

Here, $\partial_{x_j} e^{(ix\cdot\xi)} = \underbrace{i\xi_j}\, e^{ix\cdot\xi}$, so we would like a representation

$$f(x) = \int_{\mathbb{R}^n} \underbrace{a(\xi)}_{\text{FT of } f} e^{ix\cdot\xi}\, d\xi.$$

__Def__: Let $f: \mathbb{R}^n \to \mathbb{C}$ be integrable, i.e., $f \in L^1(\mathbb{R}^n; \mathbb{C})$. Then the __Fourier transform__ of f is the function

$$\hat{f}: \mathbb{R}^n \to \mathbb{C} \text{ given by}$$

$$\hat{f}(\xi) = \mathcal{F}[f](\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx.$$

__Lemma__: Let $f \in L^1(\mathbb{R}^n; \mathbb{C})$. Then:

(1) $\mathcal{F}[f(\cdot + h)](\xi) = e^{i\xi\cdot h}\, \mathcal{F}[f](\xi)$, $h \in \mathbb{R}^n$

(2) $\mathcal{F}[e^{ix\cdot\eta} f](\xi) = \mathcal{F}[f](\xi - \eta)$, $\eta \in \mathbb{R}^n$

(3) $\mathcal{F}[f(\lambda\,\cdot)](\xi) = \lambda^{-n}\, \mathcal{F}[f]\left(\frac{\xi}{\lambda}\right)$

(4) $\sup_{\xi \in \mathbb{R}^n} |\mathcal{F}[f](\xi)| \le (2\pi)^{-n/2} \|f\|_{L^1(\mathbb{R}^n)}$

Proof: (1) $\mathcal{F}[f(\cdot+h)](\xi)$

$$= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x+h)\, dx; \quad y = x+h$$

$$= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \underbrace{e^{-i(y-h)\cdot\xi}}_{e^{-iy\cdot\xi}\, e^{ih\cdot\xi}} f(y)\, dy$$

$$= e^{+ih\cdot\xi} (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-iy\cdot\xi} f(y)\, dy = e^{ih\cdot\xi}\, \mathcal{F}[f](\xi).$$

(2) + (3) = HW

(4) $$\sup_{\xi\in\mathbb{R}^n} |\mathcal{F}[f](\xi)| \le \sup (2\pi)^{-n/2} \int_{\mathbb{R}} |\underbrace{e^{-ix\cdot\xi}}_{\le 1} f(x)|\, dx$$

$$\le \sup (2\pi)^{-n/2} \|f\|_{L^1(\mathbb{R})}$$

Def: The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ consists of all smooth functions $f: \mathbb{R}^n \to \mathbb{C}$ satisfying

$$\sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta f(x)| < \infty \quad \text{for every multi-index } \alpha, \beta \in \mathbb{N}_0^n.$$

ie, $x^\alpha = (x_1^{\alpha_1} \cdots x_n^{\alpha_n})$, $\partial^\beta = \dfrac{\partial^{|\beta|}}{\partial_{\beta_1}\cdots\partial_{\beta_n}}$. In other words, all derivatives of Schwartz functions have arbitrary good polynomial decay.

Rk: (i) $f \in \mathcal{S}(\mathbb{R}^n) \iff \sup_{x\in\mathbb{R}^n} (1+|x|)^N |\partial^\beta f(x)| < \infty$, any $N, \beta \in \mathbb{N}_0^n$.

(ii) Clearly, $C_c^\infty(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n)$. Also, $e^{-a|x|^2}$, $a>0$, is Schwartz.

<u>Lemma</u>: Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then

(1) $\mathcal{F}[\partial_{x_j} f](\xi) = i\xi_j \mathcal{F}[f](\xi)$

(2) $\mathcal{F}[x_j f](\xi) = i\partial_{\xi_j} \mathcal{F}[f](\xi)$

(3) $\mathcal{F}[f \circ M](\xi) = |\det(M)|^{-1} \mathcal{F}[f]((M^{-1})^t \xi)$ for M invertible.

<u>Proof</u>: (1) $\mathcal{F}[\partial_{x_j} f](\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \partial_{x_j} f(x)\, dx$

$$\underset{\substack{\text{uses decay} \\ \text{of } f + \text{IBP}}}{=} -(2\pi)^{-n/2} \int_{\mathbb{R}^n} \underbrace{\partial_{x_j} e^{-ix\cdot\xi}}_{-i\xi_j} f(x)\, dx = i\xi_j \mathcal{F}[f](\xi).$$

(2) $\mathcal{F}[x_j f](\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \underbrace{e^{-ix\cdot\xi} x_j}_{i\frac{\partial}{\partial \xi_j}(e^{-ix\cdot\xi})} f(x)\, dx$

$$= i\frac{\partial}{\partial \xi_j} \underbrace{\left((2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx\right)}_{\mathcal{F}[f](\xi)}$$

(3) HW.

<u>Corollary</u> The Fourier Transform maps $\mathcal{S}(\mathbb{R}^n)$ to itself.

<u>Proof</u>: (Sketch) Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then,

$$\sup_{\xi\in\mathbb{R}^n} |\partial_{\xi_j} \hat{f}(\xi)| \underset{\text{by (2)}}{=} \sup_{\xi\in\mathbb{R}^n} |\mathcal{F}[x_j f](\xi)| \le \sup_{\xi\in\mathbb{R}^n} (2\pi)^{-n/2} \int_{\mathbb{R}^n} \underset{\frac{(1+|x|)^{n+1}}{(1+|x|)^{n+1}}}{|x_j f(x)|}\, dx$$

Now, f is Schwartz, then $\sup_{x\in\mathbb{R}} |(1+|x|)^N \partial^\beta f(x)|$.

$$\le (2\pi)^{-n/2} \underbrace{\int_{\mathbb{R}^n} \frac{1}{(1+|x|)^{n+1}}\,dx}_{<\infty} \cdot \underbrace{\sup_{x\in\mathbb{R}^n} (1+|x|)^{n+1}|x|\,|f(x)|}_{<\infty}. \qquad \blacksquare$$

Eg: For any $a>0$, the Gaussian $\mathbb{R}^n \ni x \longmapsto e^{-a|x|^2}$ is Schwartz and

$$\mathcal{F}[e^{-|\cdot|^2}][\xi] = (2a)^{-n/2} e^{-\frac{1}{4a}|\xi|^2}.$$

Proof: Reduce to 1D case:

$$\mathcal{F}[e^{-a|x|^2}](\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \underbrace{e^{-ix\cdot\xi}}_{e^{-i(x_1\xi_1+\dots+x_n\xi_n)}} \underbrace{e^{-a|x|^2}}_{e^{-a(x_1^2+\dots+x_n^2)}}\,d\xi$$

$$= \prod_{j=1}^{n} (2\pi)^{-1/2} \left(\int_{\mathbb{R}} e^{-ix_j\xi_j} e^{-ax_j^2}\,dx_j\right)$$

In the 1D case:

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-iy\eta} e^{-ay^2}\,dy \overset{\text{complete square}}{=} (2\pi)^{-1/2} \int_{\mathbb{R}} \exp\left[-\left(a^{1/2}y + i\frac{\eta}{2a^{1/2}}\right)^2 - \frac{\eta^2}{4a}\right] dy$$

$$\underset{\substack{\uparrow \\ \text{change of variables}}}{=} e^{-\frac{\eta^2}{4a}} \cdot \frac{1}{(2\pi)^{1/2}} \cdot \frac{1}{a^{1/2}} \int_{\mathbb{R}} \exp\left[\left(-y + i\frac{\eta}{2a^{1/2}}\right)^2\right] dy.$$

Claim: $\int_{\mathbb{R}} e^{-y^2}\,dy = \sqrt{\pi}$. Proof: $\left(\int_{\mathbb{R}} e^{-y^2}dy\right)^2 = \left(\int_{\mathbb{R}} e^{-y^2}dy\right)\left(\int_{\mathbb{R}} e^{-x^2}dx\right)$

$$= \int_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dx\,dy \underset{\substack{\uparrow \\ \text{polar coordinates}}}{=} \int_0^\infty \int_0^{2\pi} e^{-r^2} r\,d\theta\,dr = 2\pi \int_0^\infty \underbrace{e^{-r^2} r}_{\frac{d}{dr}\left(-\frac{1}{2}e^{-r^2}\right)}\,dr = 2\pi \cdot \frac{1}{2} = \pi \checkmark$$

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Eg. (cont.)

$$\mathcal{F}[e^{-a|\cdot|^2}](\xi) = (2a)^{-\frac{n}{2}} e^{-\frac{1}{4a}|\xi|^2}, \quad \xi \in \mathbb{R}^n$$

Proof: Brought it down to

$$\mathcal{F}[e^{-ay^2}](\xi) = \frac{1}{(2\pi a)^{1/2}} e^{-\frac{\xi^2}{4a}} \underbrace{\int_{\mathbb{R}} \exp\left[-\left(y + \frac{i\xi}{2a}\right)\right] dy}$$

$$\text{claim} = \int_{\mathbb{R}} e^{-y^2} dy = \sqrt{\pi}$$

Proof of claim: —

Proposition: (Fourier inversion) Let $f \in \mathcal{S}(\mathbb{R}^n)$, we have:

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}} e^{ix\cdot\xi} \hat{f}(\xi) d\xi.$$

$\rightarrow$ The Fourier adjoint $\mathcal{F}^*$ is such that $\langle \mathcal{F}[f], g \rangle = \langle f, \mathcal{F}^*[g] \rangle$, and

$$\mathcal{F}^*[g] = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} g(\xi) d\xi \quad \left(= \mathcal{F}[g](-x) \right).$$

We introduce: $\check{g}(x) = \mathcal{F}^{-1}(\mathcal{F}[g])(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} g(\xi) d\xi.$

In light of the proposition and the calculation we have:

$$\boxed{\mathcal{F}^* = \mathcal{F}^{-1}}$$

Proof of Prop: Let $f \in \mathcal{S}(\mathbb{R}^n)$, let $x \in \mathbb{R}^n$. Let $I(x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{f}(\xi) d\xi$

$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-iy\cdot\xi} f(y) dy d\xi.$ We introduce a regularization by inserting a Gaussian $e^{-\varepsilon|\xi|^2}$.

$$I(x) \overset{\text{LDCT}}{=} \lim_{\varepsilon\to 0} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-\varepsilon|\xi|^2} \underbrace{\hat{f}(\xi)}_{\shortparallel} d\xi.$$

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-iy\cdot\xi} f(y)\,dy.$$

$$\overset{\text{Fubini}}{=} \lim_{\varepsilon\to 0} \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n_y} f(y) \underbrace{\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n_\xi} e^{i(x-y)\cdot\xi} e^{-\varepsilon|\xi|^2} d\xi}_{\mathcal{F}\left[e^{-\varepsilon|\xi|^2}\right](-(x-y))} dy$$

$$= \frac{1}{(2\varepsilon)^{n/2}} e^{-\frac{1}{4\varepsilon}|x-y|^2}$$

$$= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} \underbrace{\frac{1}{(4\pi\varepsilon)^{n/2}} e^{-\frac{1}{4\varepsilon}|x-y|^2}}_{K_\varepsilon(x-y)} f(y)\,dy$$

like mollifiers

$$= \lim_{\varepsilon\to 0} (K_\varepsilon * f)(x); \quad K_\varepsilon(x) = \left(\frac{1}{\varepsilon^{1/2}}\right)^n K\left(\frac{x}{\varepsilon^{1/2}}\right), \quad \overbrace{K(z) = \frac{1}{(4\pi)^{n/2}} e^{-\frac{1}{4}|z|^2}}.$$

$$\int_{\mathbb{R}} K(z)\,dz = 1.$$

$$\Rightarrow \underbrace{\lim_{\varepsilon\to 0} (K_\varepsilon * f)(x) = f(x)}_{\text{Check!!}}$$

$f = \check{\hat{f}} \Rightarrow \mathcal{F}: \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ is a bijection

Lemma: Let $f, g \in \mathcal{S}(\mathbb{R}^n)$. Then:

(1) $\displaystyle\int_{\mathbb{R}^n} \overline{\hat{f}(\xi)}\, g(\xi)\, d\xi = \int_{\mathbb{R}^n} \overline{f(x)}\, \underbrace{\check{g}(x)}_{\mathcal{F}^{-1}[g](x)}\, dx$ → direct computation

(2) (Parseval's relation)

$$\int_{\mathbb{R}^n} \overline{f(x)}\, g(x)\, dx = \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)}\, \hat{g}(\xi)\, d\xi, \text{ or}$$

$$\langle f, g \rangle = \langle \mathcal{F}[f], \mathcal{F}[g] \rangle.$$

Proof: $\displaystyle\int_{\mathbb{R}^n} \overline{f(x)}\, g(x)\, dx = \int_{\mathbb{R}^n} \overline{f(x)}\, \check{\hat{g}}(x)\, dx = \int_{\mathbb{R}^n} \overline{\hat{f}(\xi)}\, \hat{g}(\xi)\, d\xi.$ ∎

(3) (Plancherel's identity):

$$\|f\|_{L^2_x} = \|\hat{f}\|_{L^2_\xi}.$$

Proof: $\|f\|_{L^2_x} = \|\hat{f}\|_{L^2_\xi}$. ∎

(4) (Convolution and products:)

$$\mathcal{F}[fg](\xi) = \frac{1}{(2\pi)^{n/2}} (\hat{f} * \hat{g})(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{f}(\xi - \eta)\, \hat{g}(\eta)\, d\eta$$

$$\mathcal{F}[f * g](\xi) = (2\pi)^{n/2} \hat{f}(\xi)\hat{g}(\xi). \rightarrow \boxed{HW}.$$

Proof: $\displaystyle\mathcal{F}[f \cdot g](\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, \underbrace{g(x)}_{= \mathcal{F}^{-1}[\hat{g}](x)}\, dx =$

$$= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) \left(\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot\eta}\, \hat{g}(\eta)\, d\eta \right) dx$$

$$= \int_{\mathbb{R}^n_\eta} \frac{1}{(2\pi)^{n/2}} \underbrace{\left(\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n_x} e^{-ix\cdot(\xi-\eta)} f(x)\,dx \right)}_{\hat{f}(\xi-\eta)} \hat{g}(\eta)\,d\eta$$

$\Rightarrow$ convolution

The Heat Equation:

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Consider: $\begin{cases} \partial_t u - \Delta u = 0 & \text{in } (0,\infty) \\ u(0,\cdot) = g(\cdot) & \text{in } U \end{cases}$

– For $U \subseteq \mathbb{R}^n$ open (subject to additional boundary conditions)

Goal: Obtain representation formula for IVP

$$(*) \begin{cases} \partial_t u - \Delta u = 0 & \text{in } (0,\infty) \times \mathbb{R}^n \\ u(0,\cdot) = g(\cdot) & \text{in } \mathbb{R}^n \end{cases}$$

Formal derivation: Suppose u is a smooth solution to $(*)$ in $(0,\infty)\times\mathbb{R}^n$ and $u(t,\cdot) \in \mathcal{S}(\mathbb{R}^n)$ $\forall t > 0$, and $g \in \mathcal{S}(\mathbb{R}^n)$.

Apply Fourier: $\mathcal{F}[\partial_t u](\xi) - \mathcal{F}[\Delta u](\xi) = 0$; $\mathcal{F}[u(0,x)] = \mathcal{F}[g(x)] = \hat{g}(\xi)$.

$$\mathcal{F}[\Delta u](\xi) = \sum_{j=1}^{n} \mathcal{F}[\partial_{x_j}^2 u](\xi) = \sum_{j=1}^{n} i\xi_j \, i\xi_j \, \hat{u}(t,\xi)$$

$$= -\sum_{j=1}^{n} \xi_j^2 \, \hat{u}(t,\xi)$$

$$= -|\xi|^2 \hat{u}(t,\xi).$$

Thus, $\partial_t \hat{u}(t,\xi) = -|\xi|^2 \hat{u}(t,\xi)$; $\hat{u}(0,\xi) = \hat{g}(\xi)$, so $\forall \xi \in \mathbb{R}^n$ (fix!)

$$\begin{cases} \partial_t \hat{u}(t,\xi) = -|\xi|^2 \hat{u}(t,\xi) \\ \hat{u}(0,\xi) = \hat{g}(\xi) \end{cases} \to \text{This is an ODE in time for every } \xi \in \mathbb{R}^n$$

$$\Rightarrow \hat{u}(t,\xi) = e^{-t|\xi|^2} \hat{u}(0,\xi) = e^{-t|\xi|^2} \hat{g}(\xi).$$

Then, by Fourier inversion, $u(t,x) = \mathcal{F}^{-1}[\hat{u}(t,\cdot)](x)$

$$= \mathcal{F}^{-1}[e^{-t|\cdot|^2} \hat{g}(\cdot)](x).$$

$$= \mathcal{F}^{-1}[e^{-t|\cdot|^2}] * \mathcal{F}^{-1}[\hat{g}(\cdot)](x)$$

$$= \frac{1}{(2\pi)^{n/2}} \mathcal{F}^{-1}[e^{-t|\cdot|^2}] * g(x).$$

Now $\mathcal{F}^{-1}[e^{-t|\cdot|^2}](x) = \mathcal{F}[e^{-t|\cdot|^2}](-x)$

$$= \frac{1}{(2t)^{n/2}} e^{-\frac{1}{4t}|x|^2}.$$

Thus: $u(t,x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{1}{4t}|x-y|^2} g(y)\,dy.$

Def: The function:

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & t>0,\ x\in\mathbb{R}^n \\ 0 & t<0,\ x\in\mathbb{R}^n \end{cases}$$

Remark: → For an alternative derivation of $\Phi(t,x)$ based on scaling, see Evans

→ Note that $\Phi(t,x) = \left(\frac{1}{t^{1/2}}\right)^n \Psi\left(\frac{x}{t^{1/2}}\right)$; w/ $\Psi(\xi) = \frac{1}{(4\pi)^{n/2}} e^{-\frac{|\xi|^2}{4}}$.

Since $\int_{\mathbb{R}^n} \Psi(\xi)\,d\xi = 1$

We have $\int_{\mathbb{R}^n} \Phi(t,x)\,dx = 1$, and $\{\Phi(t,\cdot)\}_{t>0}$ is an approximation of the identity

→ $(t,x) \longmapsto \Phi(t,x)$ solves the heat equation in $\mathbb{R}^n$:

$$(\partial_t - \Delta)(\Phi(t,x)) = 0.$$

Theorem: Existence of solution of heat equation on $\mathbb{R}^n$.

Assume $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Define

$$u(t,x) = [\Phi(t,\cdot) * g](x), \text{ for } (t,x) \in (0,\infty)\times\mathbb{R}^n.$$

Then (i) $u \in C^\infty((0,\infty)\times\mathbb{R}^n)$

(ii) $\partial_t u - \Delta u = 0$ in $(0,\infty)$.

(iii) $\lim_{(t,x)\to(0,x)} u(t,x) = g(x)$, in particular $u \in C([0,\infty]\times\mathbb{R}^n)$

Remark: → Heat Equation smooths initial conditions $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$

→ infinite speed of propagation!

Proof (sketch): (i) & (ii): $\Phi(t,x) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{1}{4t}|x|^2}$ is infinitely differentiable in t and x w/ uniformly bounded derivatives on $(\delta, \infty) \times \mathbb{R}^n$ so

$$\Rightarrow u \in C^\infty((0,\infty)\times\mathbb{R}^n) \text{ by DCT \&}$$

$$(\partial_t u - \Delta u)(t,x) = \int_{\mathbb{R}^n} \underbrace{(\partial_t \Phi - \Delta \Phi)}_{0}(t, x-y)\, g(y)\, dy = 0.$$

(iii) Fix $x^0 \in \mathbb{R}^n$, $\varepsilon > 0$. g = cont; choose $\delta > 0$ s.t.

$$|g(x^0) - g(y)| \le \varepsilon \quad \text{when } |y - x^0| \le \delta.$$

Then: $|x - x^0| \le \frac{\delta}{2}$,

$$\int_{\mathbb{R}^n} \Phi \, dx = 1.$$

$$|u(x,t) - g(x^0)| \le \left| \int \Phi(t, x-y) [g(y) - g(x_0)]\, dy \right|$$

$$\le \int_{B(x^0,\delta)} \Phi(t, x-y) \underbrace{|g(y) - g(x^0)|}_{\le \varepsilon} dy$$

$$+ \underbrace{\int_{\mathbb{R}^n \setminus B(x^0,\delta)} \Phi(t, x-y)\, |g(y) - g(x^0)|\, dy}_{\to 0 \text{ as } t \to 0}$$

→ 0 as $t \to 0$, see [Evans, Theorem 2.3.1].

$$\begin{cases} \partial_t u - \Delta u = 0 & t>0,\ x\in\mathbb{R}^n \\ u = g & t=0,\ x\in\mathbb{R}^n \end{cases}$$

PDEs
11-15-22

Theorem (Max principle for Cauchy problem on $\mathbb{R}^n$). Suppose that $u \in C^{2,2}((0,T]\times\mathbb{R}^n)\cap C([0,T]\times\mathbb{R}^n)$ solves the heat equation and satisfies

$$|u(t,x)| \le Ae^{a|x|^2}$$

for some $A, a>0$, and all $t>0$, x. Then

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^n} u(t,x) = \sup_{x\in\mathbb{R}^n} g(x).$$

Proof: Step 1: Assume that $4aT<1$. Then there exists $\varepsilon>0$ s.t.

$4a(T+\varepsilon)<1$. Fix $y\in\mathbb{R}^n$, $\mu>0$, define

$$v(t,x) = u(t,x) - \underbrace{\frac{\mu}{(T+\varepsilon-t)^{n/2}}\exp\left(\frac{|x-y|^2}{4(T+\varepsilon-t)}\right)}_{(4\pi)^{n/2}\mu\,\Phi(-(T+\varepsilon-t),\,x-y)}$$

Then, by calculation using $(\partial_t-\Delta)\Phi=0$,

$(\partial_t - \Delta)v = 0$ in $(0,T]\times\mathbb{R}^n$. Fix $r>0$, and let $U=B(y,r)$, $U_T = B(y,r)\times(0,T]$. U_T is bounded, so by max principle on bounded domains,

$$\sup_{U_T} v = \sup_{\Gamma_T} v.$$

First, look at initial time, $v(0,x) = g(x) - \frac{\mu}{(T+\varepsilon)^{n/2}}\exp\left(\frac{|x-y|^2}{4(T+\varepsilon)}\right)$.

Boundary values: $v(t,x) = u(t,x) - \frac{\mu}{(T+\varepsilon-t)^{n/2}}\exp\left(\frac{|x-y|^2}{4(T+\varepsilon-t)}\right)$

$$\le Ae^{ax^2} - \frac{\mu}{(T+\varepsilon)^{n/2}}\exp\left(\frac{|x-y|^2}{4(T+\varepsilon)}\right).$$

$$\le Ae^{a(|y|+r)^2} - \frac{\mu}{(T+\varepsilon)^{n/2}}e^{\frac{r^2}{4(T+\varepsilon)}}$$

By assumption $4a(T+\varepsilon) < 1$, letting $r \to \infty$, for any $x \in \mathbb{R}^n$

$v(t,x) \to -\infty$ on the spatial boundary. Thus, for $r>0$, large enough, we have

$$\max_{(t,x)\in\Gamma_T} v(t,x) \le \max_{t=0} v(t,x) \le \sup_{x\in\mathbb{R}^n} u(0,x) = \sup_{x\in\mathbb{R}^n} g(x).$$

Hence, for all $y \in \mathbb{R}^n$ and $0 \le t \le T$, $v(t,y) \le \max_{(t,x)\in U_T} v(t,x) \le \max_{(t,x)\in\Gamma_T} v(t,x)$ $\le \sup_{x\in\mathbb{R}^n} g(x)$. Thus, by definition of v,

$$\sup_{(t,y)\in[0,T]\times\mathbb{R}^n} u(t,y) \le \sup_{(t,y)\in[0,T]\times} v(t,y) + \frac{\mu}{\varepsilon^{n/2}}.$$

Now take $\mu \to 0$.

Step 2: If $4Ta < 1$ fails, repeatedly apply argument on intervals $[0, \frac{1}{8a}]$, $[\frac{1}{8a}, \frac{2}{8a}]$, ___ $\blacksquare$

Corollary: Let g be continuous on $\mathbb{R}^n$, $f \in C([0,T]\times\mathbb{R}^n)$. Then there exists at most 1 solution $u \in C^{1,2}((0,T]\times\mathbb{R}^n) \cap C([0,T]\times\mathbb{R}^n)$ of the IVP:

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } (0,T]\times\mathbb{R}^n \\ u = g & \text{on } t=0,\ x\in\mathbb{R}^n \end{cases}$$

which satisfies the growth estimate $|u(t,x)| \le Ae^{a|x|^2}$, $A, a > 0$.

Theorem (smoothness): Let $U \subseteq \mathbb{R}^n$ be open, $T > 0$. Suppose that $u \in C^{1,2}([0,T]\times U)$ solves

$$\partial_t u - \Delta u = 0 \text{ in } (0,T]\times U$$

Then $u \in C^\infty((0,T]\times U)$.

Proof: [Evans 2.3.8] Recall: $C(x,t,r) = \{(y,s) : |x-y| < r,\ t - r^2 \le s \le t\}$.

Fix $(x_0,t_0) \in U_T$, $r > 0$ small so that $C := C(x_0,t_0,r) \subseteq U_T$.

$$C' := C(x_0, t_0, \tfrac{3r}{4})$$

$$C'' := C(x_0, t_0, \tfrac{r}{2})$$

Choose a smooth cut-off function ζ s.t.

$$\begin{cases} 0 \le \zeta \le 1 & \text{on } C' \\ \zeta = 0 & \text{in a neighborhood of parabolic boundary of } C, \Gamma \end{cases}$$

Extend $\zeta \equiv 0$ on $\mathbb{R}^n \setminus C$.

PDEs
1-14-22

Def: For $\mathcal{U} \subseteq \mathbb{R}^n$ open and $T>0$, we define

$$\mathcal{U}_T := (0,T] \times \mathcal{U} \ ; \quad \Gamma_T := \overline{\mathcal{U}_T} \setminus \mathcal{U}_T .$$

Call $\mathcal{U}_T$ the parabolic cylinder, Γ_T the parabolic boundary

Theorem: (Maximum principle for heat equation) Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open and bounded. Let $T>0$. Let $u \in C^{1,2}(\mathcal{U}_T) \cap C(\overline{\mathcal{U}_T})$ be a solution to

$$\partial_t u - \Delta u = 0 .$$

Then
$$\max_{(t,x) \in \overline{\mathcal{U}_T}} u(t,x) = \max_{(t,x) \in \Gamma_T} u(t,x)$$

Proof: Let $\varepsilon > 0$, and set $v(t,x) = u(t,x) + \varepsilon |x|^2$. Suppose $v(t,x)$ is maximal at $(t_0, x_0) \in \mathcal{U}_T$. Then,

$$\partial_t v(t_0, x_0) \geq 0 ; \quad \Delta v(t_0, x_0) \leq 0 .$$

But $\underbrace{\partial_t v - \Delta v}_{\geq 0} = \underbrace{\partial_t u - \Delta u}_{0} - \varepsilon \cdot \Delta(|x|^2) = -2\varepsilon n < 0$. Thus a contradiction. Then:

$$\max_{(t,x) \in \overline{\mathcal{U}_T}} u(t,x) \leq \max_{(t,x) \in \overline{\mathcal{U}_T}} v(t,x) = \max_{(t,x) \in \Gamma_T} v(t,x) \leq \max_{(t,x) \in \Gamma_T} u(t,x) + \varepsilon \max_x |x|^2 .$$

Since $\varepsilon > 0$ was arbitrary, $\max_{(t,x) \in \overline{\mathcal{U}_T}} u(t,x) = \max_{(t,x) \in \Gamma_T} u(t,x)$. ∎

Corollary: Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open and bounded, $T>0$. Let $g \in C(\Gamma_T)$, $f \in C(\mathcal{U}_T)$. Then, there exists at most one solution to

$$\begin{cases} \partial_t u - \Delta u = f \\ u|_{\Gamma_T} = g . \end{cases}$$

Theorem: Suppose $u \in C^{1,2}((0,T] \times \mathbb{R}^n) \cap C([0,T] \times \mathbb{R}^n)$ solves:

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } (0,T] \times \mathbb{R}^n \\ u(0,\cdot) = g(\cdot) & \text{in } \mathbb{R}^n \end{cases}$$

and satisfies $u(t,x) \leq Ae^{a|x|^2}$, $0 \leq t \leq T$, $x \in \mathbb{R}^n$ for some $A, a > 0$. Then

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^n} u = \sup_{x \in \mathbb{R}^n} g.$$

Corollary: Let $g \in C(\mathbb{R}^n)$, $f \in C([0,T] \times \mathbb{R}^n)$. Then there exists at most one solution u to the problem:

$$\begin{cases} \partial_t u - \Delta u = f \\ u(0,\cdot) = g(\cdot) \end{cases}$$

satisfying $|u(t,x)| \leq Ae^{a|x|^2}$, $0 \leq t \leq T$, $x \in \mathbb{R}^n$

Theorem:

Recall $u_t - \Delta u = 0$ in $U_T = (0,T) \times U$. Claim that $u \in C^\infty$

$C = C(x,t,r) = \{ (y,s) \text{ s.t. } \|y-x\| < r,\ t-r^2 < s \le t \}$.

$C' = C(x,t,\frac{3}{4}r)$

$C'' = C(x,t,\frac{r}{2})$; $0 \le \zeta \le 1$, $\zeta \equiv 1$ on C', $\zeta \in C_c^\infty(C)$.

$v := \zeta \cdot u$ is defined on the whole space by 0 outside C.

$(-\infty, T) \times \mathbb{R}^n$

$v_t = \zeta_t u + \zeta u_t$; $\partial_i \partial_i v = \partial_i (u \partial_i \zeta + \zeta \partial_i u)$

$= u \partial_i \partial_i \zeta + 2 \partial_i u \partial_i \zeta + \zeta \partial_i \partial_i u$.

$\Rightarrow \partial_t v - \Delta v = \zeta_t u - \Delta \zeta u - 2 \nabla u \cdot \nabla v + \underbrace{\zeta(\partial_t u - \Delta u)}_{0}$

$= (\partial_t \zeta - \Delta \zeta) u - 2 \nabla u \nabla v =: \tilde{f}$.

$= 0$ on C'

$\tilde{v}(t,x) := \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) \tilde{f}(y,s)\, dy\, ds$

$v, \tilde{v}$ both are 0 at $t=0$, and both satisfy

$(\partial_t - \Delta) w = \tilde{f}$, both have growth less than Ae^{ax^2} at ∞

$\Rightarrow v = \tilde{v}$ by uniqueness of solutions.

On C'', we have $u(t,x) = v(t,x)$ (b.c. $\zeta = 1$)

$= \tilde{v}(t,x)$

$= \iint_C \Phi(x-y, t-s) \Big[(\zeta_s - \Delta \zeta)(s,y) u(s,y) - 2 \nabla u \cdot \nabla \zeta(s,y) \Big]\, ds\, dy$.

$= \iint_{C \setminus C'} \Big[\Phi(x-y, t-s) \big[\zeta_s + \Delta \zeta \big](s,y) + 2 D_y \Phi(x-y, t-s) \cdot D\zeta(s,y) \Big] u(y,s)\, dy\, ds$

$= \iint \underbrace{\underbrace{K(x,t,y,s)}\, u(y,s)\,dy\,ds}_{C^\infty\text{-smooth b.c. } K \text{ is smooth.}}$

$C^1|C^1$ defined; C^∞ smooth.

<u>Wave Equation:</u>

$$\begin{cases} -\partial_t \partial_t u + \Delta u = 0 & \text{in } (0,\infty)\times\mathbb{R}^n \\ u(0,x) = f(x) \\ \partial_t u(0,x) = g(x) \end{cases} \qquad (WE)$$

$u_{tt} = \Delta u.$

Solving wave equation using Fourier Transform:

Assume u solves W.E.; take Fourier transform of u in space on only x-variables:

$$\begin{cases} -\partial_t\partial_t \hat{u}(t,\xi) + \mathcal{F}[\Delta u] = 0. \\ \hat{u}(0,\xi) = \hat{f}(\xi) \\ \partial_t \hat{u}(0,\xi) = \hat{g}(\xi) \end{cases} \Rightarrow \quad \partial_t\partial_t \hat{u}(t,\xi) = -|\xi|^2 \hat{u}(t,\xi)$$

$x'' = -\omega^2 x$

$\rightarrow x(t) = x_0 \cos(\omega t) + v_0 \dfrac{\sin(\omega t)}{\omega}$

$$\Rightarrow \hat{u}(t,\xi) = \hat{f}(\xi)\cos(t|\xi|) + \frac{\hat{g}(\xi)\cdot\sin(t|\xi|)}{|\xi|}$$

$$\Rightarrow u(t,x) = \left(f * \mathcal{F}^{-1}[\cos(t|\xi|)]\right)(x) + \left(g * \mathcal{F}^{-1}\left[\frac{\sin(t|\xi|)}{|\xi|}\right]\right)(x).$$

[1] $\displaystyle \mathcal{F}^{-1}[\cos(t|\xi|)\hat{f}(\xi)](x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ix\xi}\left(\frac{e^{it\xi}+e^{-it\xi}}{2}\hat{f}(\xi)\right)d\xi$

$\displaystyle = \frac{1}{2}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left(e^{i(x+t)\xi} + e^{i(x-t)\xi}\right)\hat{f}(\xi)\,d\xi = \frac{1}{2}\left(f(x+t) + f(x-t)\right)$

$$\partial_s \left(\mathcal{F}^{-1}\left[\frac{\sin(|\xi| s)}{|\xi|} \hat{g}(\xi) \right](x) \right)$$

$$= \mathcal{F}^{-1}\left[\cos(|\xi| s)\, \hat{g}(\xi) \right](x).$$

$$= \frac{g(x+s)+g(x-s)}{2}$$

$$\lim_{s \to 0} \mathcal{F}^{-1}\left[\frac{\sin(|\xi| s)}{|\xi|} \right] = 0 \text{ uniformly for all } x \in \mathbb{R}.$$

$$\Rightarrow \mathcal{F}^{-1}\left(\frac{\sin(|\xi| t)}{|\xi|} \hat{g}(\xi) \right) = \int_0^t \partial_s \left(\frac{\sin(|\xi| s)}{|\xi|} \hat{g}(\xi) \right)(x)\, ds$$

$$= \int_0^t \frac{g(x+s)+g(x-s)}{2}\, ds = \int_{x-t}^{x+t} \frac{g(y)}{2}\, dy$$

$$\Rightarrow u(t,x) = \frac{1}{2}\left(f(x+t) + f(x-t) \right) + \frac{1}{2} \int_{x-t}^{x+t} g(y)\, dy.$$

PDEs
11-22-22

Linear Wave Equation:

$$\begin{cases} \partial_t^2 u - \Delta u = 0 \\ (u, \partial_t u)\big|_{t=0} = (f, g) \end{cases}$$

; Assume $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $u(t,x)$ is smooth with $u(t,\cdot) \in \mathcal{S}(\mathbb{R}^n)$.

$$\Rightarrow \partial_t^2 \hat{u} = -|\xi|^2 \hat{u}$$

$$\Rightarrow \hat{u}(t,\xi) = A(\xi)\cos(t|\xi|) + B(\xi)\sin(t|\xi|)$$

$$\Rightarrow \hat{u}(t,\xi) = \cos(t|\xi|)\hat{f}(\xi) + \frac{\sin(t|\xi|)}{|\xi|}\hat{g}(\xi).$$

<u>Theorem</u>: Let $f \in C^2(\mathbb{R})$, $g \in C^1(\mathbb{R})$. For $t \geq 0$, $x \in \mathbb{R}$, set

$$u(t,x) = \frac{1}{2}\left(f(x-t) + f(x+t)\right) + \frac{1}{2}\int_{x-t}^{x+t} g(y)\,dy. \text{ Then} \quad \boxed{1D}$$

$u \in C^2([0,\infty) \times \mathbb{R})$ is unique solution.

<u>Proof</u>: · Existence → good by direct computation
· Uniqueness → next week.

<u>Remarks</u>: · Smoothness of solution depends on regularity of data, i.e. no smoothing effect.
· Has finite speed of propagation.
· When $g = 0$:

$n=3$ For $f \in$ Schwartz $(\mathbb{R}^3)$, define spherical mean of f over the sphere of radius $t > 0$ by

$$M_t(f)(x) = \frac{1}{4\pi t^2}\int_{\partial B(x,t)} f(y)\,dS(y) \underset{y = x + t\omega}{=} \frac{1}{4\pi}\int_{S^2} f(x + t\omega)\,dS(\omega).$$

<u>Lemma</u>: If $f \in \mathcal{S}(\mathbb{R}^3)$, $t > 0$, then $M_t(f) \in \mathcal{S}(\mathbb{R}^3)$, and the map $t \longmapsto M_t(f)$ is infinitely differentiable in t, and each t-derivative is Schwartz.

<u>Proof</u>: [Stein-Shakarchi].

<u>Lemma</u>: For $f \in \mathcal{S}(\mathbb{R}^3)$ $t > 0$, $\mathcal{F}[M_t(f)(\cdot)](\xi) = \frac{\sin(t|\xi|)}{t|\xi|}\hat{f}(\xi)$.

Proof: Let's fix $\xi \in \mathbb{R}^3 \setminus \{0\}$. Then

$$\mathcal{F}[M_t(f)(\cdot)](\xi) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-ix\cdot\xi} M_t(f)(x)\,dx$$

$$= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-ix\cdot\xi} \frac{1}{4\pi} \int_{S^2} f(x+t\omega)\,dS(\omega)\,dx$$

$$= \frac{1}{2}\frac{1}{(2\pi)^{5/2}} \int_{S^2} \left(\int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x+t\omega)\,dx \right) dS(\omega)$$

$$= \frac{1}{2}\frac{1}{(2\pi)^{5/2}} \int_{S^2} \left(\int_{\mathbb{R}^3} e^{-i(x-t\omega)\cdot\xi} f(x)\,dx \right) dS(\omega)$$

$$= \frac{1}{2}\frac{1}{(2\pi)^{5/2}} \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x) \left(\int_{S^2} e^{it\omega\cdot\xi}\,dS(\omega) \right) dx$$

For fixed $\xi \neq 0$, compute:

$\int_{S^2} e^{it\omega\cdot\xi}\,dS(\omega)$. Assume coordinates are such that $\xi = (0,0,|\xi|)$

Parameter: $\omega = (\sin(\theta)\cos(\varphi), \sin(\theta)\sin(\varphi), \cos(\theta))$

$\omega\cdot\xi = |\xi|\cos(\theta)$.

$$\int_{S^2} e^{it\omega\cdot\xi}\,dS(\omega) = \int_0^{2\pi}\int_0^{\pi} e^{it|\xi|\cos(\theta)} \sin(\theta)\,d\theta\,d\varphi$$

$$= 2\pi \int_0^{\pi} e^{it|\xi|\cos(\theta)} \sin(\theta)\,d\theta$$

$$= 2\pi \int_0^{\pi} \frac{(-1)}{it|\xi|} \frac{d}{d\theta}\left(e^{it|\xi|\cos(\theta)} \right) d\theta$$

$$\frac{2\pi}{it|\xi|}(-1)\left(e^{-it|\xi|\cdot(-1)} - e^{-it|\xi|\cdot 1}\right) = \frac{4\pi}{t|\xi|}\underbrace{\frac{1}{2i}\left(e^{+it|\xi|} - e^{-it|\xi|}\right)}_{\sin(t|\xi|)}$$

$$= \frac{4\pi}{t|\xi|}\sin(t|\xi|)$$

$$\Rightarrow \mathcal{F}[M_t(f)(\cdot)](\xi) = \frac{1}{2}\cdot\cancel{\frac{1}{2\pi}}\,\underbrace{\frac{1}{(2\pi)^{3/2}}\left(\int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x)\,dx\right)}_{\hat{f}(\xi)}\cdot\frac{4\pi}{t|\xi|}\sin(t|\xi|).$$

$$= \frac{\sin(t|\xi|)}{t|\xi|}\hat{f}(\xi). \checkmark$$

$$u(t,x) = \underbrace{\mathcal{F}^{-1}\left[\frac{\sin(t|\xi|)}{|\xi|}\hat{g}(\xi)\right]}_{t\,\mathcal{F}[M_t(g)](\xi)} + \underbrace{\mathcal{F}^{-1}\left[\cos(t|\xi|)\,\hat{f}(\xi)\right]}_{\partial_t \mathcal{F}^{-1}\left[\underbrace{\frac{\sin(t|\xi|)}{|\xi|}\hat{f}(\xi)}_{t\,\mathcal{F}[M_t(f)]}\right]}.$$

$$= t\,M_t(g)(x) + \partial_t\left(t\,M_t(f)(x)\right).$$

<u>Theorem</u>: Suppose $f \in C^3(\mathbb{R}^3)$, $g \in C^2(\mathbb{R}^3)$. Then there exists a unique solution to the initial value problem to

$$\begin{cases} \partial_t^2 u - \Delta u = 0 \\ (u, \partial_t u)\big|_{t=0} = (f, g) \end{cases}$$

given by

$$u(t,x) = \fint_{\partial B(x,t)} \left(f(y) + (\nabla f)(y)\cdot(y-x) + t\,g(y)\right) dS(y).$$

<u>$n=2$</u> $\begin{cases} \partial_t^2 u - \Delta u = 0 & \text{in } (0,\infty)\times\mathbb{R}^2 \\ (u, \partial_t u)\big|_{t=0} = (f,g) & \text{w/ } f(x_1,x_2),\ g(x_1,x_2). \end{cases}$

Denote by $\partial_t^2 v - \Delta v = 0$ in $\mathbb{R}^3$ the solution to linear wave $v(t,x_1,x_2,x_3)$ w/ initial conditions $v(0,x_1,x_2,x_3) = f(x_1,x_2)$, $\partial_t v(0,x_1,x_2,x_3) = g(x_1,x_2)$. Then,

$v(t, x_1, x_2, x_3)$ can be expressed as in Kirchoff's Formula, and for any x_3, it should solve 2D equation.

Correspondingly we compute w/ $x_3 = 0$:

$$M_t(f)(x_1, x_2, 0) = \frac{1}{4\pi} \int_{S_2} f(x_1 + t\omega_1, x_2 + t\omega_2) \, dS(\omega)$$

$$= \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} f(x_1 + t\sin(\theta)\cos(\varphi), x_2 + t\sin(\theta)\sin(\varphi)) \sin\theta \, d\theta \, d\varphi.$$

Now do change of variables, $r = \sin(\theta)$, separately for $0 \le \theta \le \frac{\pi}{2}$, $\frac{\pi}{2} \le \theta \le \pi$.

$$dr = \cos(\theta) d\theta = \sqrt{1-r^2} \, d\theta, \quad d\theta = \frac{1}{\sqrt{1-r^2}} dr$$

$$\int_0^{2\pi} \int_0^{\frac{\pi}{2}} f(\qquad) \sin(\theta) \, d\theta \, d\varphi$$

$$= \int_0^{2\pi} \int_{r=0}^{r=1} f(x_1 + tr\cos(\varphi), x_2 + tr\sin(\varphi)) \frac{r}{\sqrt{1-r^2}} \, dr \, d\varphi$$

Thus, $$M_t(f)(x_1, x_2, 0) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{1} f(x_1 + tr\cos(\varphi), x_2 + tr\sin(\varphi)) \frac{r}{\sqrt{1-r^2}} \, dr \, d\varphi.$$

$$= \frac{1}{2\pi} \int_{\substack{B(0,1) \\ \cap \\ \mathbb{R}^2}} f(x + ty) \frac{1}{\sqrt{1-|y|^2}} \, dy.$$

Hence, solution to 2D problem:

$$u(t, x_1, x_2) = v(t, x_1, x_2, 0) = \partial_t \left(t M_t(f)(x_1, x_2, 0) \right) + t \cdot M_t(g)(x_1, x_2, 0)$$

$$= \partial_t \left(t \tilde{M}_t(f)(x_1, x_2) \right) + t \tilde{M}_t(g)(x_1, x_2)$$

w/ $$\tilde{M}_t(f)(x_1, x_2) = \frac{1}{2\pi} \int_{B(0,1)} f(x + ty) \frac{1}{\sqrt{1-|y|^2}} \, dy$$

PDEs
11-29-22

2 Key properties of Wave Equation

1: No smoothing of solution

2: finite speed of propagation

Energy Method:

$\rightsquigarrow$ a way to prove a priori estimates for solutions to a PDE by multiplying the PDE by a suitable function (called the multiplier) and integrating by parts.

$\rightsquigarrow$ No systematic treatment

$\rightsquigarrow$ Robust concept

Laplace Equation:

Dirichlet problem $-\Delta u = f$, $x \in U \subseteq \mathbb{R}^n$ (#1)
$u = g$ on ∂U, C^2 boundary

Prop: Let $f \in C(\bar{U})$, $g \in C(\partial U)$. Then, there exists at most one solution $u \in C^2(\bar{U})$ to (#1)

Proof: Let u_1, u_2 be two solutions. Then $w = u_1 - u_2$ satisfies

$$-\Delta w = 0, \text{ in } U \quad w = 0 \text{ on } \partial U.$$

Multiply by w; integrate over U, and IBP

$$0 = \int_U \underbrace{(-\Delta w)w}_{\substack{-\mathrm{div}(\nabla w\, w) \\ + |\nabla w|^2}} dx = -\int_U \mathrm{div}(\nabla w\, w)\, dx + \int_U |\nabla w|^2 dx$$

$$= \underbrace{-\int_{\partial U} (\nabla w \cdot \eta) w \, dS(x)}_{0} + \int_U |\nabla w|^2 dx = \int_U |\nabla w|^2 dx$$

$\Rightarrow \quad 0 = \int_{\mathcal{U}} |\nabla w|^2 dx \Rightarrow \nabla w = 0$ everywhere, so

$w \equiv c$ in $\mathcal{U}$, but since $w = 0$ on $\partial\mathcal{U}$ and w is continuous, $w \equiv 0$ in $\overline{\mathcal{U}}$.

<u>Heat Equation</u>: Consider IVP/BVP

$$\begin{aligned} \partial_t u - \Delta u &= f \quad \text{in } \mathcal{U}_T \qquad \mathcal{U}_T = (0,T) \times \mathcal{U} \quad (\#2) \\ u &= g \quad \text{on } \Gamma_T, \; T > 0 \qquad \Gamma_T = \overline{\mathcal{U}_T} \setminus \mathcal{U} \end{aligned}$$

<u>Prop</u>: If $f \in C(\overline{\mathcal{U}_T})$, $g \in C(\Gamma_T)$. Then there exists at most 1 solution $u \in C^{1,2}(\overline{\mathcal{U}_T})$ to (#2).

<u>Proof</u>: Let $u_1, u_2 \in C^{1,2}(\overline{\mathcal{U}_T})$, let $w = u_1 - u_2$. Then

$$\partial_t w - \Delta w = 0 \quad \text{in } \mathcal{U}_T$$

$$w = 0 \quad \text{on } \Gamma_T$$

Multiply by w: $\quad \partial_t w \cdot w - \Delta w \, w = 0 \quad \Rightarrow$

$$0 = \int_{\mathcal{U}} \partial_t w \, w \, dx + \int_{\mathcal{U}} |\nabla w|^2 dx \Rightarrow 0$$

$$\partial_t \int_{\mathcal{U}} \left(\frac{w^2}{2}\right) dx + \int_{\mathcal{U}} |\nabla w|^2 dx = 0$$

$$\partial_t \int_{\mathcal{U}} \left(\frac{w^2}{2}\right) dx = - \int_{\mathcal{U}} |\nabla w|^2 dx$$

$\Rightarrow$ For any $0 \le t \le T$: $\quad \frac{1}{2} \int_{\mathcal{U}} w^2(t,x) \, dx \le \underbrace{\frac{1}{2} \int_{\mathcal{U}} w(0,x)^2 dx}_{0}$

$$\Rightarrow \quad 0 \le \frac{1}{2} \int_{\mathcal{U}} w^2(t,x) \, dx \le 0$$

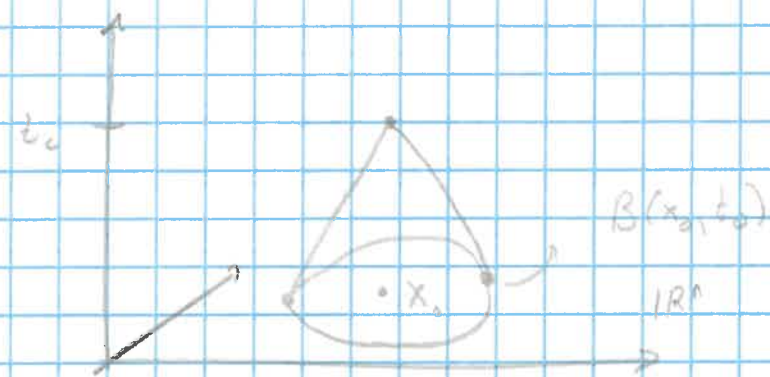
$$\Rightarrow \int_{\mathcal{U}} w^2(t,x) \, dx = 0 \Rightarrow w(t,x) = 0, \; \forall \, t, x \in \mathcal{U}_T$$

↝ Finite speed of propagation for linear wave equation

Suppose $u \in C^2([0,\infty)\times\mathbb{R}^n)$ solution to wave equation.

Fix $x_0 \in \mathbb{R}^n$, $t_0>0$. Consider the backwards light cone

$$C_{(t_0,x_0)} := \{(t,x)\in[0,\infty)\times\mathbb{R}^n \mid 0\le t\le t_0,\ |x-x_0|\le t_0-t\}$$



Prop: If $u(0,\cdot) = \partial_t u(0,\cdot) = 0$ on $B(x_0,t_0)$, then $u\equiv 0$ in $C_{(t_0,x_0)}$.

Proof: Set $e(t) := \frac{1}{2}\int_{B(x_0,t_0-t)} (\partial_t u)(t,x)^2 + |\nabla u(t,x)|^2\,dx$.

Then:
$$\partial_t e(t) = \int_{B(x_0,t_0-t)} (\partial_t u)(\partial_t^2 u)\,dx + \int_{B(x_0,t_0-t)} \overbrace{\nabla u\cdot\partial_t\nabla u}^{\mathrm{div}(\nabla u\,\partial_t u)-(\Delta u)\partial_t u}\,dx$$
$$-\frac{1}{2}\int_{\partial B(x_0,t_0-t)} (\partial_t u)^2+|\nabla u|^2\,dS(x)$$

$$= \int_{B(x_0,t_0-t)} \overbrace{\partial_t u\,\partial_t^2 u - (\Delta u)\partial_t u}^{=\partial_t u(\partial_t^2 u-\Delta u)=0}\,dx + \int_{B(x_0,t_0-t)} \overbrace{\mathrm{div}(\nabla u\,\partial_t u)}^{\int_{\partial B}(\partial_t u)(\nabla u)\cdot\nu\,dS(x)}\,dx$$
$$-\frac{1}{2}\int_{\partial B} (\partial_t u)^2+|\nabla u|^2\,dS(x)$$

Note $\left| \int_{\partial B} \partial_t u \nabla u \cdot \eta \, ds(x) \right|$

$\leq \int_{\partial B} |\partial_t u| |\nabla u \cdot \eta| \, ds(x)$

$\leq \int_{\partial B} |\partial_t u| \cdot |\nabla u| \, ds(x) \leq \int_{\partial B} \frac{1}{2} |\partial_t u|^2 + \frac{1}{2} |\nabla u|^2 \, ds(x)$

$\Rightarrow \boxed{\leq 0 \Rightarrow \partial_t e(t) \leq 0}$

Non-linear first order scalar

PDEs
12-1-22

Method of characteristics

→ U open, smooth boundary.

Let $F: U \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be smooth
$(x, z, p) \longrightarrow F(x, z, p)$

Consider $F(x, u, \nabla u) = 0$ in U, w/

$u: U \longrightarrow \mathbb{R}$ is unknown function subject to some boundary conditions; $u = g$ on Γ, $\Gamma \subseteq \partial U$
$g: \Gamma \to \mathbb{R}$ smooth.

Basic idea: Calculate $u(x)$ by finding some curve / line in U, that connects x with a point $x_0 \in \Gamma$ and along which we can compute u.
↳ Try to convert PDE into a system of ODEs, called the characteristic equations.

Basic model: Transport equation

$\partial_t u + a \partial_x u = 0 \rightarrow \begin{pmatrix} 1 \\ a \end{pmatrix} \cdot \begin{pmatrix} \partial_t u \\ \partial_x u \end{pmatrix} = 0$, i.e. directional derivative $\Rightarrow u(t,x)$ constant along the straight lines

$s \longmapsto (0, x_0) + s(1, a)$.

$x(s) = \begin{pmatrix} 0 \\ x_0 \end{pmatrix} + s \begin{pmatrix} 1 \\ a \end{pmatrix}$, $z(s) := u(x(s))$, $\frac{d}{ds} x(s) = \begin{pmatrix} 1 \\ a \end{pmatrix}$,

$\frac{d}{ds} z(s) = 0$; $\begin{cases} \frac{d}{ds} x(s) = \begin{pmatrix} 1 \\ a \end{pmatrix} \\ \frac{d}{ds} z(s) = 0 \end{cases}$

Derivation of characteristic equation:

Parametrize curve by $\vec{X}(s) = (x_1(s), \dots, x_n(s))$

$I \subset \mathbb{R} \longrightarrow \mathbb{R}^n$

Introduce $z(s) = u(\vec{x}(s))$, $\vec{p}(s) := \nabla u(\vec{x}(s))$.

$= (\partial_{x_1} u(\vec{x}(s)), \dots, \partial_{x_n} u(\vec{x}(s)))$

Assume u is C^2 solution to PDE.

Goal: choose $\vec{x}(s)$ such that we can compute $z(s)$ and $p(s)$

For this: diff $p(s)$ by s for $i = 1, \dots, n$.

$$\frac{d}{ds} p_i(s) = \sum_{j=1}^{n} \partial_{x_j} \partial_{x_i} u(\vec{x}(s)) \cdot \frac{dx_j}{ds}(s).$$

Diff F by x_i: $\quad 0 = (\partial_{x_i} F)(x, u, \nabla u) +$

$$(\partial_z F)(x, u, \nabla u)\, \partial_{x_i} u +$$

$$\sum_{j=1}^{n} (\partial_{p_j} F)(x, u, \nabla u)\, \partial_{x_j} \partial_{x_i} u. \qquad (\#2)$$

$\rightarrow$ Choose: $\frac{d}{ds} x_j(s) = (\partial_{p_j} F)(x(s), z(s), p(s)) \quad (\#3)$

Evaluate (#2) @ $x = \vec{x}(s)$, plug in (#3) then we get for $i = 1, \dots, n$

$$\frac{d}{ds} p_i(s) = -(\partial_{x_i} F)(\vec{x}(s), z(s), p(s)) - \partial_z F(\vec{x}(s), z(s), p(s)).$$

Finally: $\frac{d}{ds} z(s) = \frac{d}{ds}(u(\vec{x}(s))) = \sum_{j=1}^{n} \underbrace{(\partial_{x_j} u)(\vec{x}(s))}_{p_j} \cdot \underbrace{\frac{dx_j}{ds}}_{\partial_{p_j} F(\vec{x}(s), z(s), p(s))}$.

We arrive at the following 2n+1 ODEs; called the characteristic equations of the PDE:

(a) $\frac{d}{ds} x_j(s) = \partial_{p_j} F(\vec{x}(s), z(s), \vec{p}(s)) \quad 1 \le j \le n$

(b) $\frac{d}{ds} z(s) = \sum_{j=1}^{n} \partial_{p_j} F(\vec{x}(s), z(s), \vec{p}(s)) p_j(s)$

(c) $\frac{d}{ds} p_j(s) = -\partial_{x_j} F(\vec{x}(s), z(s), \vec{p}(s)) - \partial_z F(\vec{x}(s), z(s), p(s)) p_j(s) \quad 1 \le j \le n$

Theorem: Let $u \in C^2$ solve the PDE $F(x, u, \nabla u) = 0$ in U, assume $\vec{x}(s)$ solves (a) for $s \in I$ $\vec{x}(s) \in U$, $s \in I$ where $z(s) := u(\vec{x}(s))$. Then, $z(s)$ solves b and $\vec{p}(s)$ solves c.

Case I: $\boxed{F \text{ is linear}}$ $F = \sum_{i=1}^{n} b_i(x) \partial_{x_i} u(x) + c(x) u(x) \boxed{= 0}$

$\rightarrow F(x, z, p) = \sum_{i=1}^{n} b_i(x) p_i + c(x) z$. Then,

$\partial_{p_j} F = b_j(x) \Rightarrow \frac{d}{ds} x_j(s) = b_j(x(s))$

$$\frac{d}{ds} z(s) = \sum_{j=1}^{n} b_j(x) p_j(s)$$
$$= -c(\vec{x}(s)) z(s)$$

Observe the hierarchy: ODE for curve is closed by itself: once we know $x(s)$, we can solve for $z(s)$. So $\vec{p}(s)$ not need to determine $u(x(s)) = z(s)$.

Ex: $\begin{cases} x_1 \partial_{x_2} u - x_2 \partial_{x_1} u = u & \text{in } U; \\ u = g & \text{on } \Gamma \end{cases}$ $U = \{x_1 > 0, x_2 > 0\}$, $\Gamma = \{x_1 > 0, x_2 = 0\}$

$\rightsquigarrow$ $F(x,u,\nabla u) = \underbrace{\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}}_{\vec{b}(x)} \cdot \nabla u - u$

Thus, $\frac{d}{ds} x_1(s) = -x_2(s)$

$\frac{d}{ds} x_2(s) = x_1(s)$

$\frac{d}{ds} z(s) = -(-1) z(s) = z(s)$

$\rightsquigarrow$ $\frac{d}{ds}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$; for $(x^0, 0) \in \Gamma$,

$\Rightarrow$ $x_1(s) = x^0 \cdot \cos(s)$
$x_2(s) = x^0 \cdot \sin(s)$

$\Rightarrow$ $z(s) = z(0) e^s$; $z(0) = g(x^0)$, $z(s) = g(x^0) e^s$.

$\rightarrow$ $u(x_1, x_2) = ?$

Right now: $(s, x^0) \longrightarrow (x_1, x_2)$.

Need to <u>invert</u> this map to get $s(x_1, x_2)$
$x_0(x_1, x_2)$

Then get $u(x_1, x_2) = z(s(x_1, x_2))$.

$x^0 = (x_1^2 + x_2^2)^{1/2}$, $\tan(s) = \frac{x_2}{x_1} \Rightarrow s = \arctan\left(\frac{x_2}{x_1}\right)$.

<u>Thus</u>: $u(x_1, x_2) = z(s(x_1, x_2))$

$= g\left((x_1^2 + x_2^2)^{1/2}\right) \exp\left(\arctan\left(\frac{x_2}{x_1}\right)\right)$.

$\boxed{\text{Burgers}}$ $\partial_{x_2} u + u \partial_{x_1} u = 0$, $u = g$ on Γ

$U = \{x_2 > 0\}$

$\Gamma = \{x_2 = 0\}$

$\longrightarrow F(x, z, p) = p_2 + z p_1$

$\longrightarrow \frac{d}{ds} x_1(s) = \partial_{p_1} F = z(s)$

$\frac{d}{ds} x_2(s) = \partial_{p_2} F = 1$

$\frac{d}{ds} z(s) = \partial_{p_1} F p_1 + \partial_{p_2} F p_2 = z p_1 + p_2 = F(x, z, p) = 0$

$\to x_2(s) = x_2(0) + s = s$

$\to z(s)$ is constant, $z(s) = z(0) = g(x^0)$

$\to \frac{d}{ds} x_1 = z(s) = z(0) = g(x^0)$,

$\Rightarrow x_1 = x^0 + s g(x^0)$.