Inverse Problems and Imaging

S. Gerberding

April 3, 2026

1 Jan-17

1.1 Preliminaries and Overview

Plan of course:

- What are inverse problems?
- Well and ill posed problems
- Four fundamental spaces
- Examples of inverse problems
- Physics and math or main computed tomography modalities
- Radon and X-Ray transforms
- Fourier analysis, sampling theory
- Radon inversion formulas
- Hybrid modalities
- Examples of implementation

What is an *inverse problem*? Inputs x, *unknown* operator A outputs $y = Ax$. Find A by sending many inputs and observing outputs, ie, black box. This is more or less nonsense put this way: often need more information.

Questions!

1. Does a solution exist?
2. Is the data sufficient for a unique solution?
3. Stability wrt errors, parameters?
4. Numerical algorithms.

Definition 1.1. *A problem is well-posed if it has properties 1-3.*

Sad fact: *practically all inverse problems are ill-posed.* But some of them are not that bad.

1.2 Linear Algebra Recollections

Let A be a matrix, linear operator; $Ax = y$. When you know x, A you know what to do. Given any x, and knowing y, you can find A. If you know A and y, A needs to be invertible (and all the ways that is possible).

In the case of diagonal matrix, if the entries are decaying to 0, we have a problem, since solving them is theoretically possible, computers will have trouble because of the scale. The hard part: for many inverse problems, those diagonal entries go to zero, and sometimes quite quickly.

2 Jan-22

Well-posed problem:

1. There exists a solution
2. Solution is unique
3. Stability of solution with respect to data

Vector Space V:

- $v_1, v_2 \in V \implies v_1 + v_2 \in V$
- $\lambda \in \mathbb{C}, v \implies \lambda v \in V$

Definition 2.1. *V, W vector spaces, $A : V \to W$, mapping=operator=transformation. A linear mapping is a mapping A that preserves the structure of a vector space: for all $v, w \in V$, $\lambda \in K$, $A(\lambda v + w) = \lambda Av + Aw$.*

Definition 2.2. *A basis $v_1, \ldots, v_n \in V$ is a set of vectors such that for any $w \in V$, $w = \lambda_1 v_1 + \cdots + \lambda_n v_n$, and this representation/linear combination is unique. The number of basis vectors is called the dimension of the vector space.*

Definition 2.3. *Euclidean space (or Hilbert space) is a vector space equipped with an inner product (scalar product, dot product) which is a function $v_1, v_2 \mapsto \lambda$; $(v_1, v_2), v_1 \cdot v_2, \langle v_1, v_2 \rangle$ such that:*

1. *Linear w.r.t. the v_1 variable: $(\lambda_1 v_2 + \lambda_2 v_2, w) = \lambda_1(v_1, w) + \lambda_2(v_2, w)$*
2. *Anti-symmetry: $(v_1, v_2) = \overline{(v_2, v_1)}$*
3. *$(v, v) \geq 0$ and $(v, v) = 0$ iff $v \equiv 0$.*

We can now define $\|v\| := \sqrt{(v_1, v_2)}$—the norm—and orthogonality: $v_1 \perp v_2$ iff $(v_1, v_2) = 0$.

Definition 2.4. *If A is linear and $(Av, Aw) = (v, w)$, then A is unitary or orthogonal.*

Polarization formula: If you know norms are preserved, then the inner product is preserved.
Suppose A, y are known, and we want to solve $Ax = y$, $A : V \to W$.

1. Solution exists: if $R(A) \neq W$, a solution may not exist, but perhaps you can get close
2. Solution is unique: if $N(A) := \{x : Ax = 0\}$ is such that $N(A) \neq \{0\}$, then nothing is ever unique: all you have to do is take $z \in N(A)$, then $A(x + z) = Ax = y$, hence solution is not unique. If $N(A)$ is nontrivial, then A is an *isomorphism* between V and W.

3 Jan-24

Euclidean space: (w, v), operator A.

Definition 3.1. *Let H be a Euclidean space. Let $A : H \to H$ be a linear operator. The adjoint of A, denoted A^*, a map such that $(v, A^*w) = (Av, w)$ for all $v, w \in H$. If $A = A^*$, A is self-adjoint.*

Proposition 3.1. *If A is self-adjoint, then A has eigenvectors composing a basis of the domain.*

4 Jan-29

4.1 What is SVD?

For $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, there exists U, O such that:

$$A = U\Sigma O$$

where U, O are unitary, and Σ is diagonal (even for non-square scenarios) with entries $\sigma_i \geq 0$ and is decreasing.

Another thing: if

$$A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & -0 & 10^{-30} \end{pmatrix}$$

5 is the norm of the matrix, $5, 0.1$ is the closest rank 2 operator, etc.

A rule of thumb about singular values of operators of functions: $A : \Omega \to \mathcal{S}$, A "adds k derivatives" with $\dim(\mathcal{S}) = d$, then $\sigma_n\ \frac{1}{n^{k/d}}$.

4.2 What is harmonic analysis?

The idea of using *symmetries*, eg, rotations, shifts. Mathematically, if you have two operator A, B and they commute, then you have a symmetry.

Lemma 4.1. *Let A, B be matrices and let B be diagonalizable with distinct eigenvalues. If A and B commute, then A is diagonalizable with the same eigenvectors, though perhaps different eigenvalues.*

Proof. Exercise. □

Example: $A = \frac{\mathrm{d}}{\mathrm{d}x}$, $B =$ shift:

$$ABf = \frac{\mathrm{d}}{\mathrm{d}x} f(x+s) = f'(x+s)\frac{\mathrm{d}x}{\mathrm{d}x} = f'(x+s) = BAf.$$

4.3 Fourier Analysis

Big idea: any constant coefficient linear differential operator commutes with shifts.

5 Jan-31

5.1 Harmonic Analysis as using symmetries

For operator A, B is a symmetry of A if $AB = BA$. The idea is that if you find symmetry that is nice, and can find a good basis, use it. Recall from last time that if B has distinct eigenvalues, then the eigenvectors of B are also eigenvectors of A.

5.2 Commutativity

5.3 Fourier Analysis-linear algebra

Let V_n be a FD Hilbert space. Let $e_1, \ldots, e_n$ be an orthonormal basis of V_n. Then, for all $f \in V_n$, we can produce *Fourier coefficients*, $f_j := (f, e_j)$. The map $f \mapsto \{f_j\}_1^n$ is called *Fourier Analysis*. Now, if we are given f_j, we can find f by:

$$f = \sum_1^n f_j e_j.$$

Also:

$$\|f\|^2 = \sum_1^n |f_j|^2.$$

Using Polarization:

$$(f, g) = \sum_1^n f_j g_j$$

5.4 Eigenfunctions of shifts

A shift is $f(x) \mapsto f(x+a)$, or $S_a f(x) := f(x+a)$. Looking for f such that it is an eigenfunction of *all* shifts, so that $S_a(f) = \mu_a f$. One example: $f(x) = e^{\lambda x}$, so $S_a(f) = e^{\lambda a} e^{\lambda x}$. Indeed, these are the *only* ones.

5.5 Fourier Series

What Fourier did, we narrow down our set to *periodic* functions. For simplicity, we take 2π as our period, $[0, 2\pi]$, so $f(x + 2\pi) = f(x)$. So on this set, we require $e^{\lambda(x+2\pi)} = e^{\lambda x}$, but this only works when $e^{2\pi\lambda} = 1$, so $\lambda = ni$. Thus we look at $e_n(x) := e^{nix}$. Now we need an inner product: $(f, g) := \int_{-\pi}^{\pi} f(x)\overline{g(x)}\ \mathrm{d}x$—so we're looking in $L^2([-\pi, \pi])$. Let us check that our basis is o.n.:

$$\int e_n e_m\ \mathrm{d}x = \int_{-\pi}^{\pi} e^{inx} e^{-imx}\ \mathrm{d}x = \int_{-\pi}^{\pi} e^{i(n-m)x}\ \mathrm{d}x$$

This is 0 when $n \neq m$ and $n = m$ yields $\|e_n\|^1 = 2\pi$. So we correct:

$$e_n(x) = \frac{1}{\sqrt{2\pi}} e^{nix}.$$

It can be shown that for all $f \in L^2$, $f = \frac{1}{2\pi} \sum_{-\infty}^{\infty} f_n e^{inx}$. So, for $f \in L^2(-\pi, \pi)$, we have $f \mapsto \{f_j\}_{-\infty}^{\infty}$ with

$$f_j := \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(x) e^{-inx}\ \mathrm{d}x.$$

Also:

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} f_n e^{inx}$$

which converges in L^2. The same formulas hold from linear algebra. The big picture: if your problem is symmetric with *shifts*, use Fourier analysis. If you use Fourier basis, it will diagonalize your operator! For example, derivatives:

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \frac{1}{\sqrt{2\pi}} \sum in f_n e^{inx}$$

Thus derivative is diagonalized with Fourier expansion. For example, if your problem is rotationally invariant, use Fourier series because rotations are just shifts on e^{ix}. But if you're working on $\mathbb{R}$, we're in trouble since we assumed periodic functions.

6 Feb-5

6.1 Fourier series

Suppose $f' \in L^2$. Then, $f' = \frac{1}{\sqrt{2\pi}} \sum (in) f_n e^{inx}$, and so $\sum |n|^2 |f_j|^2 < \infty$. This puts a stronger condition on the function. So differentiating makes functions less smooth—makes then look worse. On the flip side, integration makes functions more smooth. But this causes the diagonal of the operator to be n^{-k}, and thus appears our singular value problem.

6.2 Smoothness

6.3 Smoothing operators

6.4 Larger period

To from $[-\pi,\pi]$, we can go to $[-k\pi,k\pi]$. We just check when $e^{\lambda x}$ is periodic. We get: $e^{\lambda(x+2k\pi)} = e^{2\pi k\lambda}e^{\lambda x}$, so when $e^{2\pi k\lambda} = 1$ that is when $k\lambda = in$, so $\lambda = \frac{in}{k}$. We we have $e^{i\frac{n}{k}x}$. This leads to denser frequencies. When $k \to \infty$, the frequencies go nuts. So we take $\frac{n}{k} =: \xi$. So, when we want to go to the whole line, we think of $e^{i\xi x}, \xi \in \mathbb{R}$.

6.5 Fourier Transform

We thus define:

$$\widetilde{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} f(x)e^{-i\xi x}\ \mathrm{d}x. \tag{6.1}$$

If $f \in L^1(\mathbb{R})$, this is legit. For $L^2(\mathbb{R})$, it will converge in the following sense:

$\widetilde{f}_n(x) = \frac{1}{\sqrt{2\pi}}\int_{-N}^{N} f(x)e^{-i\xi x}\ \mathrm{d}x$ will converge (Finite measure). Now, as $N \to \infty$, $\widetilde{f}_N(x) \to \widetilde{f}(\xi) \in L^2$. Then we have the Fourier Transform:

$$f(x) \mapsto \widetilde{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{-i\xi x}\ \mathrm{d}x.$$

and inverse:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widetilde{f}(\xi)e^{i\xi x}\ \mathrm{d}\xi.$$

Note:

$$\|f\|_{L^2} = \|\widetilde{f}\|_{L^2}$$

so Fourier transform is unitary.

In $\mathbb{R}^n$, things are the same. For $f \in L^2$. Define

$$\widetilde{f}(\xi) = \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} f(x)e^{-i\xi x}\ \mathrm{d}x$$

7 Feb-7

Easy to show:

$$\mathcal{F}[\mathcal{F}[(f)(x)]] = f(-x).$$

For shifts: $f(x) \to f(x+a) := T_a(f)(x)$. What would be $\widetilde{T_a(f)}(\xi)$? We would get

$$\widetilde{T_a(f)}(\xi) = e^{i\xi a}\widetilde{f}(\xi).$$

Also:

$$\widetilde{\partial_x f}(\xi) = i\xi\widetilde{f}(\xi) \tag{7.1}$$

Dilation:

$$\widetilde{f}_t(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(ty)e^{-i\xi y}\ \mathrm{d}x = \widetilde{f}(\xi/t)t^{-n}$$

Rotation:

$$f_A(x) = f(Ax)$$

then

$$\widetilde{f_A}(\xi) = \int_{\mathbb{R}} f(Ax)e^P - \xi \cdot x \, \mathrm{d}x = \int_{\mathbb{R}^n} f(y)e^{iA\xi \cdot y} \, \mathrm{d}y = \widetilde{f}(A\xi)$$

Convolution:

$$(f * g)(x) = \int f(x-y)g(y) \, \mathrm{d}y = (g * f)(x)$$

There is no function such that $g * f = f$ for all nice f. Observe:

- $g(x) = 0$ for all $x \neq 0$.
- $\int g(x) \, \mathrm{d}x = 1$.

Def:
singular support $f = \{x : f \text{ is not smooth}\}$.

Definition 7.1. *f is smooth at (or near) x_0 if there exists a C^∞ function ϕ such that ϕf is C^∞ and $\phi(x_0)$.*

8 Feb-12

8.1 Fourier Transform

$f(x)$ on $\mathbb{R}^n$, $f \in L^2(\mathbb{R}^n)$. We define:

$$\widetilde{f}(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi} \, \mathrm{d}x$$

Example: $e^{-x^2/2}$ satisfies $\frac{\mathrm{d}G}{\mathrm{d}x} + xG = 0$ and $\frac{\mathrm{d}\widetilde{G}}{\mathrm{d}\xi} + \xi\widetilde{G} = 0$. Thus, $\widetilde{G} = ce^{-\xi^2/2}$. It turns out that $c = 1$.
Also remember: $f_r(x) = f(rx)$, then $\widetilde{f}_r = r^{-1}\widetilde{f}(\xi/r)$.
Another example: box function.

$$\Pi_1(x) = \begin{cases} 1 & |x| \leq 0.5 \\ 0 & \text{else} \end{cases}$$

Also have $\Pi_1 := \Pi_1 * \Pi_1$.
What is its Fourier transform? Can show:

$$\widetilde{\Pi_1} = \frac{i}{\sqrt{2\pi}} \frac{\sin(0.5\xi)}{\xi}.$$

8.2 Properties

Convolution: The following holds (not too hard to show, use $x - y + y = x$ and $x = x - y$ to get

$$\widetilde{f * g}(\xi) = \sqrt{2\pi}\widetilde{f}\widetilde{g}.$$

Tensor product: $f(x), g(y) \mapsto f(x)g(y)$. Then:

$$\widetilde{f(x)g(y)} = \widetilde{f}(\xi_1)\widetilde{g}(\xi_2).$$

8.3 Paley-Wiener theorem

Theorem 8.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$. Then, supp$(f) \subset [-a, a]$ iff $\widetilde{f} : \mathbb{C} \to \mathbb{C}$ is entire and $|\widetilde{f}(\xi)| \leq ce^{a|\Im(\xi)|}$.*

Sketch. □

8.4 Band-Limit

Definition 8.1. *If supp$(\widetilde{f}) \subset [-b, b]$, then f is b-bound limited.*

8.5 Sampling

Theorem 8.2. *If f is b-bound limited and $h \leq \pi/b$, then f is uniquely determined by its values at the points kh. (Here, we sample at intervals with steps h.)*

The formula:

$$f(x) = \sum_k f(kh)\,\mathrm{sinc}(\pi/h(x-kh)).$$

8.6 Distributions

9 Feb-14

From last time: $e^{-x^2/2} = ce^{-\xi^2/2}$. How to know $c = 1$?

Fun fact: if $f(x)$ is even, then $\widetilde{f}$ is real.

If f is b-band limited, then $\mathrm{supp}(\widetilde{f}) \subset [-b, b]$.

π/b is the [...] frequency: if the discretization $h < \pi/b$, then we can find all of f.

Recall:

$$\mathrm{sinc}(x) := \begin{cases} \frac{\sin(x)}{x} & x \neq 0 \\ 1 & x = 1 \end{cases}$$

If you know values at kh, you can get f via:

$$f(x) = \sum_k f(kh)\,\mathrm{sinc}(\frac{\pi}{h}(x-kh)).$$

9.1 Distributions

Consider $H(x)$, the heavyside function. What would be $\partial_x H$? What would happen?

$$\int_{\mathbb{R}} \partial_x H f(x)\ \mathrm{d}x = \underbrace{H(x)f(x)|_{i\infty}^{\infty}}_{0} - \int_{\mathbb{R}} H(x)\partial_x f(x)\ \mathrm{d}x = f(0)$$

Of course, the first expression doesn't mean anything since H doesn't have a derivative. So, what we do instead is define a *distribution*, in this case we call it the δ_0 distribution (commonly called the δ function). It is a linear functional on C_0^∞ functions, and is defined via:

$$\langle \delta, f\rangle := f(0)$$

We can push this to higher dimensions: if S is a surface in $\mathbb{R}^n$, then we can define δ_S via:

$$\langle \delta_S, f\rangle := \int_S f(x)\ \mathrm{d}s$$

What would be $\langle \delta', f\rangle$?

$$\langle \delta', f\rangle = \int \delta' f\ \mathrm{d}x = -\int \delta f'\ \mathrm{d}x = -f'(0).$$

We can define the Fourier Transform of a distribution:

$$\widetilde{\delta}(\xi) = \frac{1}{\sqrt{2\pi}}\langle \delta, e^{-ix\xi}\rangle \text{``}=\text{''} \frac{1}{\sqrt{2\pi}}\int \delta(x)e^{-ix\xi}\ \mathrm{d}x$$

Definition 9.1 (Micro-local)**.** *Let f be a function and x_0 be a point. We say that f is smooth near $x_)$ if there exists $\phi \in C_0^\infty$ with $\phi(x_0) \neq 0$ such that $\phi f \in C_0^\infty$.*

Definition 9.2 (Singular-support)**.**

$$\mathit{singsupp}(f) := \{x : f \text{ is not smooth}\}$$

Definition 9.3 (Micro-local smooth)**.** *We say that f is micro-locally smooth at (x_0, ξ_0) if there is a $\phi \in C_0^\infty$, $\phi(x_0) \neq 0$ such that $\widetilde{\phi f}(\xi)$, a cone exists around ξ_0 so that $|\widetilde{f\phi}| \leq \frac{C_N}{(1+|\xi|)^N}$, and the cone must be none-degenerate. The wave-front is:*

$$WF := \{(x_0, \xi_0) : f \text{ is not m-l smooth}\}.$$

10 Feb-19

10.1 Distribution, convolution

Operations on distributions:

- Linear operators
- Shift:
$$\psi(x) \to \psi(x-a), \langle \psi_a, f\rangle := \langle \psi, f(\cdot + a)\rangle.$$
- Multiplication: if ψ is a distribution, g test function, $\langle \psi g, f\rangle = \langle \psi, gf\rangle$.

Hilbert Transform: Let ϕ be a distribution, f test function. We have $\phi * f$ as an object. The *Hilbert Transform* is:

$$Hf = \frac{1}{\pi} p.v. \frac{1}{x} * f = \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{f(x)}{x-y} \, \mathrm{d}x$$

Also can show:

$$\widetilde{\frac{1}{x}}(\xi) = \begin{cases} -i & \xi > 0 \\ i & \xi < 0 \end{cases}$$

10.2 Sobolev Spaces + WF

$$H^k(\mathbb{R}^n) := \{f(x) : D^\alpha f \in L^2(\mathbb{R}^n), \ |\alpha| \leq k\}.$$

with

$$\|f\|_{H^k}^2 = \sum_{|\alpha|\leq k} \|D^\alpha f\|_{L^2}^2 = \int_{\mathbb{R}^n} \sum_{|\alpha|\leq k} |D^\alpha f(x)|^2 \, \mathrm{d}x = \int \sum_{|\alpha|\leq k} |\xi|^{2\alpha} |\widetilde{f}(\xi)|^2 \, \mathrm{d}x = \int_{\mathbb{R}^n} \left(\sum_{|\alpha|\leq k} |\xi|^{2\alpha} \right) |\widetilde{f}(\xi)|^2 \, \mathrm{d}\xi$$

Note that we can change the polynomial in ξ to either $(1+|\xi|)^{2k}$ or $(1+|\xi|^2)^k$, since these would be equivalent in the sense $C_1 p_1(\xi) \leq p_2(\xi) \leq C_2 p_1(\xi)$, and would give equivalent norms.

But what this means is that a function in H^k must have Fourier transform that decays faster than polynomials of $2k$ degree.

10.3 Tomography

10.4 ODE example

10.5 Radon Transform

11 Feb-21

11.1 Old Tomography

The old idea is to use geometry, and slide the souce of x-rays and the films so that one slice of the object was weighted, while the others were distorted.

11.2 CT: Dimensions, collimation, geometries

Collimator: help to only pick up one direction of data. (Avoid noise from other outside signal.)

Geometry relates to the way in which beams are sent. Beams could be parallel, could be from a "cone" type source.

11.3 An ODE

$$\frac{\mathrm{d}I}{\mathrm{d}x} = -\mu(x)I(x)$$

Nice: separable variables.

$$I(x) = Ce^{-\int_0^x \mu(x)\ \mathrm{d}x}$$

with $C = I(0)$. Then, $I(1) = I(0) = e^{-\int_0^1 \mu(x)\ \mathrm{d}x}$. (Can do for other values than 0 and 1.)

$$\ln \frac{I(0)}{I(1)} = \int_{[a,b]} \mu(y)\ \mathrm{d}y.$$

11.4 Beer's Law

Tells you what happens if there is a beam of photons passing through x in a certain direction. Let $I(x)$= intensity of beam at point x. $\Delta I = -\mu(x)\Delta x I$; $\mu(x)$ tells you how much is absorbed. So μ is what we are after! It tells us density! So knowing values of I, we can find (average) of μ!

Now in theory, we suppose we can send any L through the object. Then: if we known $\int_L \mu(x)\ \mathrm{d}x$ for all L. So, the *Radon Transform* is the map that takes μ to all line integrals. We want to invert this transform to capture μ.

$\mu(x)$ is compactly supported, piece-wise smooth function in $\mathbb{R}^2$, and $R\mu(L) = \int_L \mu(x)\ \mathrm{d}x$. We need to introduce coordinates to make sense of this problems.

Normal coordinates: take origin. The line L can be defined by two parameters: a unit vector and the distance from the origin. How to write the integral?

$$\int_{x \cdot w = s}$$

is okay, but not so nice still. Consider $\omega^\perp$ (orthogonal to normal vector). Then, $x = s\omega + t\omega^\perp$. So instead:

$$\int_{x\cdot\omega} \mu(x)\ \mathrm{d}x = \int_{\mathbb{R}} \mu(s\omega + t\omega t^\perp)\ \mathrm{d}t =: R\mu(s,\omega)$$

12 Feb-26

12.1 Electronic Book

12.2 Notations

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be compactly supported in the unit disk. (Can extend to L^2.) For a line L:

$$f(x) \mapsto g(L) := \int_L f(x)\ \mathrm{d}l$$

We introduce coordinates we parameterize L with $(s, w) \in \mathbb{R} \times S^1$, with $w = (\cos(\phi), \sin(\phi))$. Thus, we define:

$$(Rf)(s,\omega) = \int_{x\cdot\omega=s} f(x)\ \mathrm{d}l = \int_{-\infty}^{\infty} f(s\omega + t\omega^\perp)\ \mathrm{d}t.$$

If $g = Rf$, then $g(-s, -\omega) = g(s, \omega)$.

Now, define again

$$\widetilde{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-ix\cdot\xi}\ \mathrm{d}x$$

$$g(s) \mapsto \widetilde{g}(\sigma) := \int_{\mathbb{R}} g(s)e^{-s\sigma}\ \mathrm{d}s$$

Radon transform:

$$f(x) \mapsto Rf(s,\omega) = g(s,\omega) = \int_{\mathbb{R}} f(w\omega^{\perp} + s\omega)\ \mathrm{d}t$$

1. $g(s,\omega) = g(-s,-\omega)$
2. Shifts: $f(x) \to f_a(x) = f(x+a)$, with $a = (a\cdot\omega^{\perp})\omega^{\perp} + (a\cdot\omega)\omega$. Then, $Rfa = \int_{\mathbb{R}} f((t + a\omega^{\perp} + (s + (a\cdot\omega)\omega))\ \mathrm{d}t = \int_{\mathbb{R}}(ft' + \omega^{\perp} + (s + (a\cdot\omega)\omega)\ \mathrm{d}t' = g(s + (a\cdot\omega),\omega)$. Thus, shifting the function shifts the Radon Transform. So Fourier is appealing.
3. Rotations: Let A be an orthogonal transformation. Denote $f_A(x) = f(Ax)$. Now, $Rf_A = \int_{x\cdot\omega=s} f(Ax)\ \mathrm{d}l = \int_{A^{-1}y\cdot\omega} f(y)\ \mathrm{d}l = \int_{y\cdot A\omega} f(y)\ \mathrm{d}l$. Hence, $f(Ax) \mapsto g(s, A\omega)$.
4. Dilation: $x \mapsto rx$, $r > 0$. $f_r(x) := f(rx)$; look at $Rf_r = \int_{x\cdot\omega=x} f(rx)\ \mathrm{d}l = \int_{y\cdot\omega=rs} f(y)\frac{1}{r}\ \mathrm{d}y$ take $y = rx$. Hence $Rf_r(s,\omega) = \frac{1}{r}Rf(rs,\omega)$.

12.3 X-ray, radon, divergent beam

12.4 Invariance

12.5 Mellin Transform

Look at two groups: $(\mathbb{R}^+,\cdot)$ and $(\mathbb{R},+)$. Both are commutative. These are the same group, using the exponential (or log depending on direction). This is the idea of Mellin transform.

12.6 Projection-slice theorem

Take fourier series of $g = Rf$. Then, with $y = t\omega^{\perp} + s\omega$, so $s = y\cdot\omega$, $\sigma\omega = \xi$:

$$\widetilde{g}(\sigma,\omega) = \int_{\mathbb{R}} g(s,\omega)e^{-s\sigma s}\ \mathrm{d}s = \int\int f(t\omega^{\perp}+s\omega)e^{-i\sigma s}\ \mathrm{d}t\ \mathrm{d}s = \int\int f(y)e^{-(}\ \mathrm{d}y = \int\int f(y)e^{-iy\xi}\ \mathrm{d}y = \widetilde{f}(\sigma\omega).$$

Hence $Rf = g(s,\omega) \mapsto_{\mathcal{F}} \widetilde{f}(\sigma\omega)$. So now just invert Fourier, and we recover f!

12.7 Ridge Functions

12.8 Back projection

13 Feb-28

13.1 Melin transform

13.2 X-ray, Radon, divergent beam

$f(x) \to g(s,\omega) := \int_{x\cdot\omega=s} f(x)\ \mathrm{d}l = \int_{\mathbb{R}} f(t\omega^{\perp} + s\omega)\ \mathrm{d}t$

Divergent beam transform, another way of writing x ray transform: x, θ, $\int_{\mathbb{R}} f(x + t\theta)\ \mathrm{d}t$.

13.3 Back projection

13.4 Projection-slice, Fourier-slice

$g(s) \to \widetilde{g}(\sigma) := \int e^{is\sigma} g(s)\ \mathrm{d}s$.

$f(x) \to g(s,\omega) = Rf(s,\omega) \to \widetilde{g}(\sigma,\omega)$, and we actually have $\widetilde{g}(\sigma,\omega) = \widetilde{f}(\sigma\omega)$.

See F. Natterer for numerical analysis work.

13.5 Inversion

14 Mar-4

14.1 Projection-slide

$f(x) \to Rf(x,\omega) = g(s,\omega)$. $\widetilde{f}(\sigma\omega) = \widetilde{g}(\sigma,\omega)$. (Fourier-slice formula)

14.2 Back projection

take $g(s,\omega) \to R^{\#} g(x) := \int_S g(x\cdot\omega,\omega)\ \mathrm{d}\omega$.

$(R^{\#}Rf)(x) = \int\int f(t\omega^{\perp} + s\omega)\ \mathrm{d}t = .?.?.? = \int \frac{2f(y)}{|x-y|})\ \mathrm{d}y = \frac{2}{|x|} * f(x)$.

14.3 Hilbert Transform

$(Hu)(t) = \frac{1}{\pi} p.v. \int \frac{u(s)}{s-t}\ \mathrm{d}s$ Filtered back projection: $f(x) = \frac{1}{4\pi} R^{\#} H \frac{\mathrm{d}}{\mathrm{d}s} Rf$. Equivalent formulation: ρ-filtered back projection $\sqrt{-\Delta} R^{\#} Rf$.

Take $f \in C_0^{\infty}$, $Rf = g(s,\omega) \in C_0^{\infty}(T)$, $T = S \times \mathbb{R}$. $\omega \in S, s \in \mathbb{R}$. Then $\widetilde{f}(\sigma\omega) = \widetilde{g}(\sigma,\omega)$. These are smooth in σ,ω; but when are they smooth in x,y? Well, we need

Theorem 14.1. *Let $g(\sigma,\omega) \in C_0^{\infty}(T)$ and $f := f(\sigma\cdot\omega)$. For $f \in C_0^{\infty}(\mathbb{R}^2)$ it is necessary and sufficient that $g(\sigma,\omega) = g(-\sigma,-\omega)$; $\forall k = 0,1,\ldots$, if $G_k(\omega) = \int t^k g(t,\omega)\ \mathrm{d}t$ then G_k is a homogeneous polynomial of degree k w.r.t ω.*

14.4 Inversion

14.5 Polar coord.-Range

15 Mar-6

15.1 Radon Transform/ Fourier slice

$\widetilde{f}(\sigma\omega) = \widetilde{g}(\sigma,\omega)$ To get that $\widetilde{g}$ is smooth in cartesian coordinates at 0, we need the following:

1. $g(s,\omega) = g(-s,-\omega)$
2. $G_K(\omega) = \frac{\partial^k}{\partial\sigma^k}\widetilde{g}(\sigma,\omega)|_{\sigma=0}$ should be a restriction to S^1 of a homogenous poly of degree k. So we would get $G_k(\omega) = \int s^k g(s,\omega)\ \mathrm{d}s$ (homogeneous poly of degree k). The back projection:

$$R^{\#} H \frac{\mathrm{d}}{\mathrm{d}s} Rf$$

Definition 15.1. *$f \in H^2(\mathbb{R}^2)$ iff $\int |\tilde{f}(\xi)|^2 (1+|\xi|)^{2s}\ \mathrm{d}\xi < \infty$*

In our case:

$$\int |\tilde{g}(\sigma,\omega)|^2 (1+|\sigma|^2)^{2s} |\sigma|\ \mathrm{d}\sigma\ \mathrm{d}\omega < \infty$$

can modify to:

$$\int |\tilde{g}(\sigma,\omega)|^2(1+|\sigma|^2)^{2s+1/2}|\frac{\sigma}{1+|\sigma|}vert\ \mathrm{d}\sigma\ \mathrm{d}\omega < \infty$$

So we get:

$$\frac{1}{C}\|g\|_{H^{s+1/2}} \leq \|f\|_s \leq C\|g\|_{H^{s+1/2}}.$$

So we have an equivalence of norms, and so we get a closed range for the Radon Transform.

15.2 Uniqueness

15.3 Existence = Range

15.4 Stability

15.5 Transport equation

15.6 Incomplete Data