Evolutionary Games and Population Dynamics

Part 1: Logistic Growth

1.1: Population dynamics and density dependence.

Three basic situations:

1) Competition: - 2+ species ~~[illegible]~~ compete for same resource
   - The more of one species, the worse for the other
   -

2) Mutualism: - Reverse of competition. The more of one the better for the other.

3) Host-parasite: - Parasites benefit from host but do not benefit it.

1.2: Exponential Growth

Let R be the rate of growth between *discrete* generations. The equation is given by

$x' = Rx$, where x is one generation and the next.

(alternatively: $x_{n+1} = Rx_n$) $\Longleftrightarrow$ $\boxed{\frac{x_{n+1}}{x_n} = R.}$ $*$

If R remains constant, then the density after t generations is:

$R^t x$, when x is the initial population

↳ if $R>1$, this explodes.

In nature this will be checked somehow.

Now consider continuous generations. Let $x(t)$ be the population number at time t, then

$$\frac{x(t+h)-x(t)}{h}$$

is the average rate of growth in the time interval $[t, t+h]$.

→ Of course ~~[illegible]~~ $x:[0,\infty)\to\mathbb{N}$, but when $\mathbb{N}$ is large enough, individual births and deaths are, heuristically, negligible.

Assume, thus, the ~~[illegible]~~ existence of a time derivative:

$$\frac{dx(t)}{dt}=\lim_{h\to 0}\frac{x(t+h)-x(t)}{h}:=\dot{x}(t).$$

(Observe that $(\log x)^{\bullet}=\frac{1}{x(t)}\cdot\dot{x}(t)=\frac{\dot{x}}{x}$.)

Then $\frac{\dot{x}}{x}$ can be viewed as the <u>per-capita growth rate</u>, or the as the average contribution of one individual to the growth. If this rate is constant, then:

↳ ie, $\left(\frac{\dot{x}}{x}=r\right)$.

the ODE becomes:

$$\dot{X} = rx.$$

Quickly solving this:

$$\frac{dx}{dt} = rx \Rightarrow \frac{1}{x}dx = r\cdot dt \Rightarrow \int \frac{1}{x}dx = \int r\,dt \Rightarrow$$

$$\ln(x) = rt + C \Rightarrow$$

$$x = e^{rt+c} = e^{c}\cdot e^{rt}.$$

Suppose $X(0) = X_0$. Then, $X(0) = e^{c} = X_0$. Thus, $c = \ln(X_0)$. Thus, $X = e^{\ln(X_0)}e^{rt} = X_0 e^{rt}$, or

$$\boxed{X(t) = X(0)\,e^{rt}}$$

current pop   initial pop   growth rate.

1.3: Logistic Growth

As mentioned earlier, growth will be checked. (More population $\Rightarrow$ less resources $\Rightarrow$ smaller growth rate).

Basic case: rate linearly decreases as a function of X. That is, ~~[illegible]~~

$$\text{rate} = -\frac{X}{K} + r = r\left(1 - \frac{X}{K}\right).$$

when x is small, this can be approximated with exponential.

Then the ODE becomes: $\dot{X} = rX\left(1 - \frac{X}{K}\right)$.

Clearly if $X = 0$, $X = K$, then $\dot{X} = 0$. When $X < K$, $\frac{X}{K} < 1$, so $\dot{X} > 0$, and when $X > K$, $\dot{X} < 0$.

The solution is
$$X(t) = \frac{K\,X(0)\,e^{rt}}{K + X(0)(e^{rt} - 1)}.$$

1.4 The Recurrence relation $x' = Rx(1-x)$ $\left(x_{n+1} = Rx_n(1-x_n)\right)$.

Clearly, $\frac{x'-x}{x}$ gives per capita rate of increase.

($x'-x$ is change $\Rightarrow \frac{x'-x}{x}$ = change per unit)

Again, suppose the rate decreases linearly. Then we arrive at the discrete version of the logistic model:

$$y' = Ry\left(1-\frac{y}{K}\right) \quad \left(y_{n+1} = Ry_n\left(1-\frac{y_n}{K}\right)\right).$$

For practical purposes, y cannot exceed K (carrying capacity).

Suppose $\boxed{R>4}$, and y is about $K/2$. Then we would have (say $R=5$)

$$y' = 5\cdot\frac{K}{2}\left(1-\frac{1}{K}\cdot\frac{K}{2}\right) = 5\cdot\frac{K}{2}\cdot\left(\tfrac{1}{2}\right) = \frac{5}{4}K > K.$$

But this, again does not make practical sense.

More formally, if $R>4$, and $y=\alpha K$, $1>\alpha>1/2$, K carrying capacity, then

$$R>4 \Rightarrow R\alpha K > 4\alpha K > 2\alpha K \Rightarrow$$

$$R\alpha K(1-\alpha) > 2K(1-\alpha) > 2K\cdot 1/2 > K.$$

$$\Rightarrow \quad y' > K. \Rightarrow C.$$

Indeed, this holds for all $1/2 < \alpha < 1$, so $y_n < \alpha K$ $\forall\, \alpha > 1/2$, but then $y_n < \frac{K}{2}$ for all n, so, the carrying capacity is at most $\frac{K}{2} \Rightarrow C$. ■

Thus we require $0<R<4$.

Set $x=\frac{y}{K}$, $x'=y'/K$, and this gives us:

$$\underbrace{x'=Rx(1-x)}_{\text{Still discrete.}} \quad \text{(Easy to check).}$$

1.5: Stable and unstable fixed points.

Let $0<R<4$. Then, the map

$$x \longmapsto Rx(1-x) \text{ is a dynamical system}$$

on $[0,1]$ ("$K=1$" here).

Proof:

$$0 \leq (x-1/2)^2$$

$$0 \leq x^2-x+1/4$$

$$-(x^2-x+1/4) \leq 0$$

$$-x^2+x \leq 1/4$$

$$x(1-x) \leq 1/4$$

$Rx(1-x) \leq 1$. Thus, $F(x):=Rx(1-x)$ is s.t.

$F([0,1]) \subseteq [0,1]$ (invariant). ∎

Note that the earlier proof shows that, if $R>4$, then the above is not invariant under $[0,1]$.

Indeed, here is a better mathematical statement:

Theorem: Let $K \in \mathbb{R}_{>0}$, $F : \mathbb{R} \to \mathbb{R}$, $F(x) := Rx(1-\frac{x}{K})$.
Then, $F([0,K]) \subseteq [0,K]$ iff $R < 4$.

Proof:

$$0 \le \left(\frac{x}{\sqrt{K}} - \frac{\sqrt{K}}{2}\right)^2$$

$$\Updownarrow$$

$$0 \le \frac{x^2}{K} - x + \frac{K}{4}$$

$$\Updownarrow$$

$$-\frac{x^2}{K} + x \le \frac{K}{4} \qquad \text{iff } R<4$$

$$\Updownarrow \qquad\qquad \downarrow$$

$$R\left(-\frac{x^2}{K}+x\right) \le R\frac{K}{4} \le \frac{4}{4}K = K$$

$$\Updownarrow$$

$$R\left(-\frac{x^2}{K}+x\right) \le K$$

$$Rx\left(1-\frac{x}{K}\right) \le K.$$

$$0 \le Rx\left(1-\frac{x}{K}\right) \le K \leftarrow \text{iff } x \in [0,K].$$

$$0 \le F(x) \le K.$$

∎

Now suppose further on that $K=1$.
Clearly $F(0)=0$, and so 0 is a fixed point.
Suppose, then that $R \le 1$. Then,

$$F(x) = Rx(1-x) \le x(1-x) \le x.$$

Therefore, the orbit of x (or $F \circ F \circ F \circ \ldots \circ F(x)$) decreases monotonically to 0.

(Biologically, if the growth rate per individual is less than 1, then the species will go extinct; ie, if $\underbrace{\frac{\dot{X}}{X}}$ < 1, the extinction is coming.

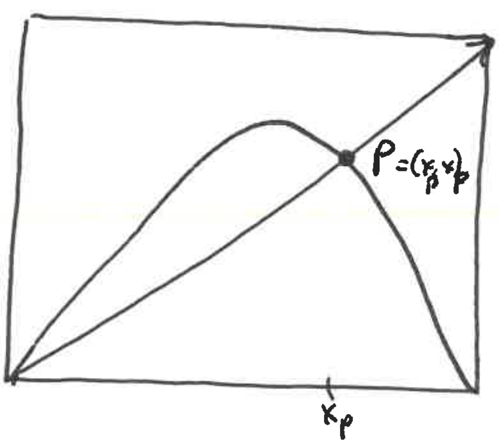
$$\frac{\dot{X}}{X} < X(1-x) \le 1-x \le 1. \Rightarrow \dot{X} \le X.$$
)

Thus, we only consider when $R>1$. Clearly F is a parabola with 0's at $x=0$, $x=1$. The max value is given by:

$$F'(x) = R - 2Rx = 0 \iff x = 1/2$$

$$\Rightarrow F(1/2) = \frac{R}{2}(\cdot 1/2) = \frac{R}{4} \Rightarrow \text{max @ } (1/2, R/4).$$

F intersects $y=x$ at a unique point p:



→ and what happens when we are close to P? Do we stay close? Well, see next section. :)

Thus, $F(x_p) = x_p$ is another fixed point. Let's find this point:

$$F(x_p) = Rx_p(1-x_p) = x_p \iff Rx_p - Rx_p^2 = x_p \iff$$
$$(R-1)x_p - Rx_p^2 = 0 \iff x_p(R-1-Rx_p) = 0 \quad (x_p = 0 \checkmark) \iff$$

$R-1-Rx_p=0 \iff x_p=\frac{R-1}{R} =: p$. This is the fixed point.

What is the stability of x_p?

By the mean value theorem, $\exists\, c \in (x, p)$ s.t.

$$\frac{F(x)-F(p)}{x-p} = F'(c) \Rightarrow \quad F(x)-F(p) = F'(c)\cdot(x-p) \Rightarrow$$

$F(x)-F(x_p) = F'(c)(x-x_p)$. Thus:

$$\frac{F(x)-F(x_p)}{x-x_p} = \frac{x-x_p}{x-x_p}F'(c) = F'(c).$$

*

$$\Rightarrow \quad \left|\frac{F(x)-F(x_p)}{x-x_p}\right| = |F'(c)|.$$ Let $x \to x_p$. Thus,

$|F'(x_p)| = |F'(c)|$. Thus, if $|F'(x_p)|<1$, then

$|F'(c)|<1$ when x (and thus c) is close to p. Then,

if you are close to p, then you get closer to p.

$|F(x)-x_p| \le |x-x_p|$. Thus, $F(x) =: x'$ is closer to x_p than x. And so, when x is close enough to x_p, then the orbit of x goes to x_p.

In this way, the fixed point is <u>asymptotically stable</u>.

On the other hand, if we had $|F'(x_p)|>1$, then $|F(x)-x_p| > |x-x_p|$, and so points would move further away. In this sense, x_p would be <u>stable</u>.

Since $F'(x) = R-2Rx$ and $x_p = \frac{R-1}{R}$, $F'(x_p) = R-2(R-1)$
$= 2-R$.

Thus when $1<R<3$, $|2-R|<1$, but not when $3<R<4$.

1.6: Bifurcations:

Let $p = x_p$. Denote $F^{(j)}(x) := \underbrace{F(F(F\ldots(F(x)}_{j\text{-times}}\ldots$

Since F is a quadratic, $F^{(2)}(x)$ is quartic, it has a local min at $x=1/2$ and two local max symmetrically to either side.

(Quick analysis: $F^2(x) = F(F(x)) \Rightarrow (F^2(x))' = F'(F(x))F'(x)$, so $1/2$ remains a critical number. We ~~only~~ need to find when $F'(F(x)) = 0$ (recall we are working w/ polynomials here.)

It is obvious that p remains a fixed point for F^2. ($F^2(p) = p$).

When $1 < R \leq 3$, (ie, when p is stable), p remains the only fixed point. (The slope at p ~~is~~ less than 1, so ~~po~~ the graph nearby p does not cross $y=x$.) When $3 < R < 4$, (or when $|F'(p)| > 1$), we get 2 more fixed points, (since the opposite is true). That is, for some p_1, p_2, $F^2(p_1) = p_1$, $F^2(p_2) = p_2$, and

$F(p_1) = p_2$, $F(p_2) = p_1$. Indeed, suppose not,

suppose $F(p_1) = p_3$. Then, $F^2(p_3) = F^3(p_1) = F(p_1)$. Thus p_3 has period 2, but only p_1 and p_2 have this property. (If $p_3 = p$, then $F(p_1) = p \Rightarrow p_1 = F(p) = p$, but $p_1 \neq p$).

Def: A point x is a periodic point for the dynamical ~~point~~ system $T: X \to X$ if $\exists K \in \mathbb{N}$ s.t. $T^K(x) = x$, where K is the smallest such number. K is the period of x.

→ Natural in discrete models: the delay in population regulation causes the population to "jump" over the fixed point: Some of the jumpings won't dampen, and simply repeat.

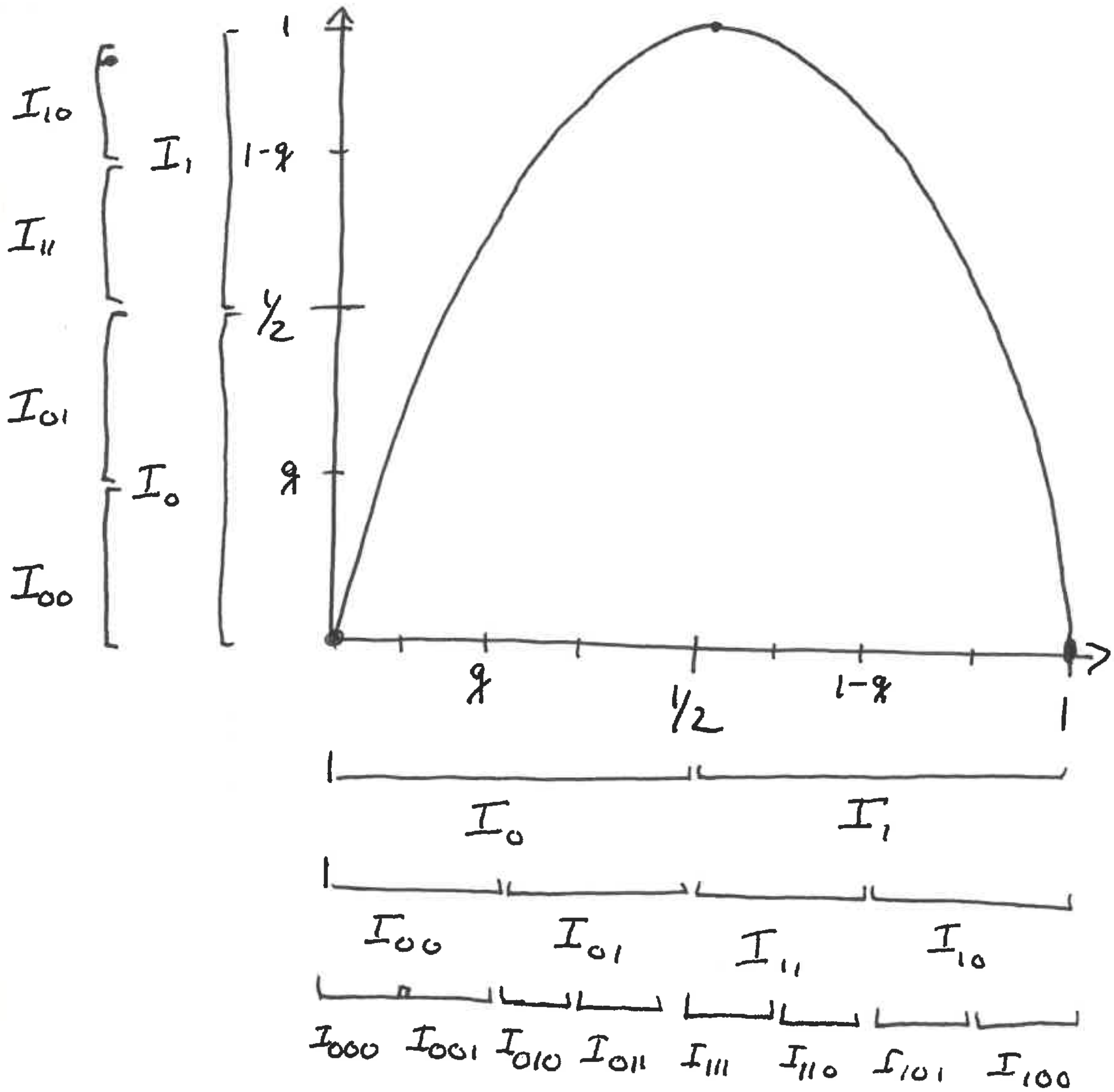
Def: A bifurcation point is a value of a parameter for which a fixed point changes from stable to unstable.

→ See YouTube for how R effects # of fixed points and their characteristics.

1.7: Chaotic Motion.

Take the F and $\boxed{R=4.}$ The max of F is 1, and $[0, \frac{1}{2}] = I_0$ and $I_1 = [\frac{1}{2}, 1]$ are both mapped bijectively to $[0,1]$ by F. We can also find $I_{00} = [0, q]$ and $I_{01} = [q, \frac{1}{2}]$ which are mapped onto I_0 and I_1 respectively. I_1 can be decomposed into $I_{10} = [1-q, 1]$, $I_{11} = [\frac{1}{2}, 1-q]$, s.t $I_{11} \xrightarrow{F} I_1$, $I_{10} \to I_0$.

* In general, the higher R is, the crazier the dynamics *



$F[I_{000}] = I_{00} \Rightarrow F^2[I_{000}] = F[I_{00}] = I_0 \Rightarrow$

$F^3[I_{000}] = F[I_0] = [0,1]$. (and similarly for the rest)

Thus, $F^j[I_\alpha] = [0,1]$ (as a bijection), for when $|\alpha| = j$.

$x \in I_{0100}$, $x \in I_0$, $F(x) \in I_1$, $F^2(x) \in I_0$, $F^3(x) \in I_0$.

That is, F "reads" I_α left to right, and sends x to that place.

However, for $x \neq y$, often can find a large enough n so that x and y will belong to different I_α, I_β (note $|\alpha| = |\beta|$), but will have vastly different paths. Thus, "computable" $\neq$ "predictable" (that is given a small error, that error will "explode" after enough iterations of F are applied). Thus even though the model is deterministic is appears random in the long run.

Also, in this respect, continuous models can be quite tame when compared to their discrete counterparts.

Part 2: Lotka-Volterra Equations for predator-prey systems

2.1: A Predator-Prey Equations

Context: Suppose we have some prey population, denoted by x, and a predator population denoted by y. Suppose that, in the absence of predators, the growth rate is constant, say

$$\frac{\dot{x}}{x} = a.$$

But when there are predators, suppose that the growth rate is negatively linear with respect to the predator population:

$$\frac{\dot{x}}{x} = a - by \implies$$

$$\dot{x} = x(a-by)$$ (That is, the more predators, the lower the growth rate.)

As for the predators, when there is no prey, we would need a negative growth rate, but when there are prey, it should increase. That is (as above)

$$\frac{\dot{y}}{y} = -c + dx \implies$$

$$\dot{y} = y(-c+dx).$$

Thus we have the system:

$$\begin{cases} \dot{x} = x(a-by) \\ \dot{y} = y(-c+dx) \end{cases}$$

2.2: Solutions of differential equations

Write $\dot{\vec{X}}(t) = \vec{F}(t, \vec{x})$ for the ODE:

$$\dot{X}_i = F_i(\underset{\text{"time"}}{t}, \underbrace{x_1, \dots, x_n}_{\text{"space"}}).$$

Expressed more:

$$\dot{\vec{X}}(t) = (\dot{X}_1(t), \dots, \dot{X}_n(t))$$
$$= (F_1(t, x_1, \dots, x_n), \dots, F_n(t, \overbrace{x_1, \dots, x_n}^{\text{these are}})),$$

t dependent...? ↓ years after ~~(2.H.)~~

where $F_i : U \subseteq \mathbb{R}^{n+1} \to \mathbb{R}$ and is continuously differentiable in all variables.

a solution is a map:

$$t \longrightarrow \vec{X}(t) = (X_1(t), \dots, X_n(t))$$

from some interval I into $\mathbb{R}^n$ such that the components $X_i(t)$ are differentiable and satisfy

$$\dot{X}_i(t) = F_i(t, X_1(t), \dots, X_n(t)) \quad \forall i \in \{1:n\} \quad \forall t \in I.$$

(Solution: $\vec{X} : \mathbb{R} \to \mathbb{R}^n$ by $\vec{X}(t) = (X_1(t), \dots, X_n(t))$ s.t.

$$\dot{X}_i(t) = F_i(t, X_1(t), \dots, X_n(t)).$$

Heuristically, at every time t, there attached at every point $\vec{x} \in \mathbb{R}^n$ an n-dimensional vector $F(t,x)$ whose components are $F_i(t, x_1, \dots, x_n)$. (View this as the wind velocity at point $\vec{x}$ at time t). A path $t \to \vec{X}(t)$ describes the motion of a particle in n-space: both position $\vec{X}(t) = (X_1(t), \dots, X_n(t))$ and velocity

$\dot{X}(t) = (\dot{x}_1(t), \dots, \dot{x}_n(t))$ at time t are n-dimensional vectors. A solution is a path whose velocity matches, at every time t, the wind velocity at the point $X(t)$.

$\vec{f}: \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$.

- $\vec{f}(t, \vec{x})$ is the vector field: You can independently shift time or space in f; that is, $\vec{x}$ is not time dependent, ~~so the~~, so there is no path.
- The solution, however, is a map $\vec{x}: \mathbb{R} \longrightarrow \mathbb{R}^d$ ~~such~~, $\vec{x}(t) = (x_1(t), \dots, x_n(t))$ such that

$$\dot{\vec{x}}(t) = (\dot{x}_1(t), \dots, \dot{x}_n(t)) = (f_1(t, \vec{x}(t)), \dots, f_n(t, \vec{x}(t)))$$

for all t. Note that $\vec{x}(t)$ is $\vec{x}$ for $\vec{f}$, but from the path. So, the path is a solution if

$$\dot{\vec{X}}(t) = \vec{f}(t, \vec{x}(t)) \text{ for all } t.$$

Def: An initial condition is the specification of the position of the particle at a given time.

Theorem: For every ~~diff~~ initial condition, there exist a unique solution to the differential equation.

Note: For our purposes, initial condition will be $t=0$, and will be at some point $\vec{x}_0$. The solution from the initial condition will be denoted by $\vec{x}(t)$. That is, $\vec{x}(0) = \vec{x}_0$. Also, for fixed, the solution depends continuously on $\vec{x}_0$.

Def: Time-independent ODEs are of the form

$$\dot{\vec{x}} = \vec{f}(\vec{x}), \text{ or } \dot{x}_i = f_i(x_1, \dots, x_n).$$

That is, the vector field does not change in time, only in space. If this is the case, then for two particles starting at $\vec{X}_0$, if the first, say $\vec{x}(t)$ is released first, then $\vec{y}(t)$ T time later, then,

$$\vec{y}(t+T) = \vec{x}(t) \text{ for all } t.$$

Def: Let $X \subseteq \mathrm{Dom}(\vec{F})$. If for all $\vec{x} \in X$, t the solution $\vec{X}(t)$ is defined and lies entirely in X, then the ODE determines a continuous dynamical system on X.

To every $\underbrace{\underset{\substack{\downarrow \\ \text{initial condition}}}{\vec{x} \in X} \text{ corresponds its orbit } \underset{\substack{\downarrow \\ \text{solution.}}}{\{\vec{X}(t) : t \in \mathbb{R}\}}}$.

The ODE "acts" on $\vec{x}$ continuously to give $\vec{x}(t)$.

There are three types of solutions may occur:

1) If $\vec{x}(t) \equiv \vec{X}_0$ for all t, then $\vec{X}_0$ is a fixed point. Note that $\vec{F}(x_0) = \vec{0}$, so no change occurs.
2) If $\vec{x}(T) = \vec{x}$ for some $T > 0$, but $\vec{x}(t) \neq \vec{x}$ for $t \in (0, T)$, then $\vec{x}$ is called a periodic point, and T is the period. Note that, $\vec{x}(T+t) = \vec{x}(t)$ for all t, thus all points are also periodic with period T.
3) If $t \to X(t)$ is injective, then the orbit never intersects itself.

2.3 Analysis of Lotka-Volterra predator-prey equation

There are 3 immediate solutions:

1) $X(t) = y(t) = 0 \Rightarrow \dot{x}(t) = 0 = \dot{y}(t)$, so 0 is fixed. (No species)
2) $X(t) = 0, \Rightarrow y(t) = y(0)e^{-ct}$ for $y(0) > 0$. (No prey)
3) $y(t) = 0 \Rightarrow X(t) = X(0)e^{at}$ for $X(0) > 0$. (No predator).

Each of these solutions has an orbit:

1) $(0,0)$ $\forall t$
2) Positive x-axis
3) Positive y-axis.

} Boundary of first quadrant.

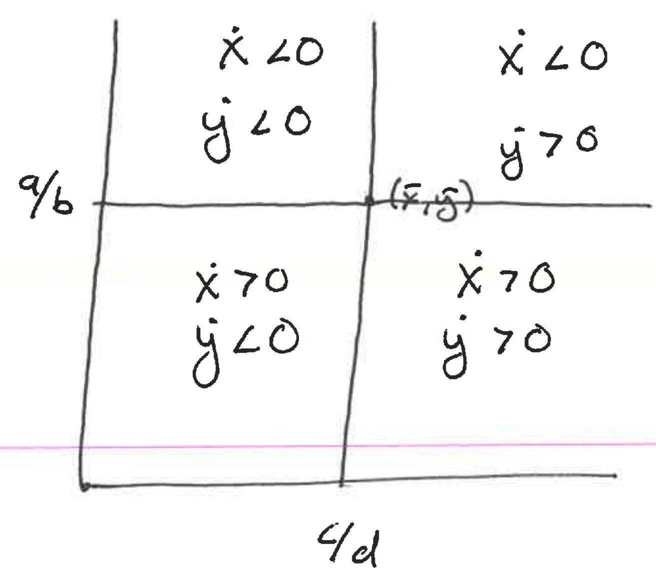
Note that $\mathbb{R}^2_+$ is invariant under the system.

We claim there is another resting point. Suppose $\bar{x}, \bar{y} \neq 0$ (so we are not on the boundary), and

$\dot{x} = 0$
$\dot{y} = 0$ at $\bar{x}, \bar{y}$. Then,

$$\left.\begin{aligned}\dot{x} &= \bar{x}(a - b\bar{y}) = 0 \\ \dot{y} &= \bar{y}(-c + d\bar{x}) = 0\end{aligned}\right. \overset{\bar{x} \neq 0 \neq \bar{y}}{\Longrightarrow} \begin{aligned} a - b\bar{y} &= 0 \\ -c + d\bar{x} &= 0 \end{aligned} \Rightarrow \underbrace{\left.\begin{aligned} \bar{y} &= a/b \\ \bar{x} &= c/d. \end{aligned}\right\}}_{\text{fixed point.}}$$

When, $y < a/b$, then $\dot{x} > 0$, and when $x > c/d$, then $\dot{y} > 0$. So we get:
↳ (and vice versa)



Now, $\dot{x} \cdot \frac{(c-dx)}{x} = (a-by)(c-dx)$ $\Rightarrow$

$$\dot{y} \cdot \frac{(a-by)}{y} = (a-by)(-c+dx)$$

$$\dot{x}\frac{(c-dx)}{x} + \dot{y}\frac{(a-by)}{y} = 0. \Rightarrow$$

$$\frac{d}{dt}\left(c\ln(x) - dx + a\ln(y) - by\right) = 0$$

Let $H(x) = \bar{x}\ln(x) - x, \quad G(y) = \bar{y}\ln(y) - y$,

$V(x,y) = dH(x) + bG(y) \Rightarrow V(x,y) = c\ln(x) - dx + a\ln(y) - by \Rightarrow$

$$\frac{d}{dt}V(x(t),y(t)) = 0 \Rightarrow V(x(t),y(t)) = C, \text{ constant}$$

That is, the function V is constant along the solution, called the constant of motion.

Since $\frac{dH}{dx} = \frac{\bar{x}}{x} - 1, \quad \frac{dG}{dy} = \frac{\bar{y}}{y} - 1,$

$$\frac{d^2H}{dx^2} = -\frac{\bar{x}}{x^2} < 0, \quad \frac{d^2G}{dy^2} = -\frac{\bar{y}}{y^2} < 0$$

we have that H has a max at $\bar{x}$, and G has a max at $\bar{y}$. Thus V is maximized at the fixed point $(\bar{x},\bar{y})$, and $V \to -\infty$ as it moves away from $(\bar{x},\bar{y})$. The level sets are closed curves around $(\bar{x},\bar{y})$. Thus, the solutions must remain on the level set, and so must return to starting point, and so are periodic.

2.4: Volterra's Principle

Prop: Let the predator-prey system be as above and let T be its period. Then, the time averages will remain constant, and ie,

$$\frac{1}{T}\int_0^T x(t)\,dt = \bar{x}\,; \quad \frac{1}{T}\int_0^T y(t)\,dt = \bar{y}.$$

Proof: Recall: $\frac{d}{dt}(\log x) = \frac{\dot{x}}{x} = a - by \implies$

$$\int_0^T \frac{d}{dt}(\log x)\,dt = \int_0^T a - by\,dt. \implies$$

$$\log(x(T)) - \log x(0) = aT - \int_0^T by\,dt \implies$$

period $\rightarrow$ $\log x(T) - \log(x(T)) = aT - \int_0^T by\,dt = 0 \implies$

$$\frac{1}{T}\int_0^T y\,dt = \frac{a}{b} = \bar{y}\text{, and likewise for } x. \quad \blacksquare$$

(Volterra's Explanation): With fishing, the rate of growth is decreased, say instead of a, we have some $a-k>0$. But, since the growth rate of the prey is smaller, (ie, fewer fish to eat), the predators growth rate is decreased: instead of c, we get $c+m$ (so $-(c+m)$ in the system). Thus,

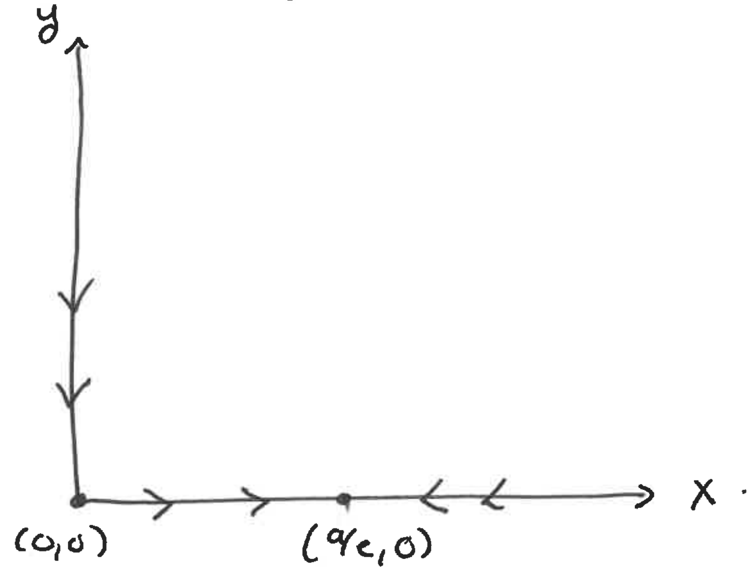
$$\bar{y} > \bar{y}_p = \frac{a-k}{b}\,; \quad \bar{x}_p = \frac{c+m}{d} > \bar{x}$$

$\bar{y}_p \rightarrow$ density in the perturbed state.

Hence, we see on average <u>fewer</u> predators and <u>more</u> prey. So, fishing actually <u>increases</u> the prey population.

Well, $\dot{y} = y(-c - fy)$, $y > 0$, $\Rightarrow$ $\dot{y} < 0$, so $y \to (0,0)$.
(as expected).

This occurs regardless of parameters. ~~...~~ :



But what happens in the plane? $(x, y > 0)$. Consider

Def: The x-isocline is ~~...~~ a line s.t.

$x \neq 0$, $\dot{x} = 0$ (or when prey does not change)

The y-isocline is ~~...~~, a line s.t.

$y \neq 0$, $\dot{y} = 0$.

Observe that if the x and y isoclines intersect at some point, then $\dot{x} = \dot{y} = 0$ at that point, and thus is a fixed point. ~~...~~

For our current purposes, if $x \neq 0 \neq y$, but $\dot{x} = \dot{y} = 0$, we have case (4) from above, ie

$ex + by = a \rightarrow x$-isocline.

$dx - fy = c. \rightarrow y$-isocline.

Thus we have that the isoclines are 2 straight lines in plane.

That is, stopping fishing would lead to more predators and fewer prey.

2.5 The predator-prey equation w/ intraspecific competition

The above model has the unrealistic property that in the absence of predators, prey will have exponential growth. (as we have seen, logistic is more realistic). Indeed, we may modify the equation by accounting for the same logistic considerations: start with

$$\dot{x} = x(a-ex) \quad (\text{logistic}),\ e>0$$

$$\dot{y} = y(-c + \cancel{\ } - fy) \quad \left(\text{logistic, but not entirely necessary, as predator population will not explode}\right).$$

$f \geq 0$.

as before, we have $\mathbb{R}^2_+$ invariant $(0,0)$, $(x^*,0)$, $(0,y)$ are all still solutions, for new equations:

$$\begin{cases} \dot{x} = x(a-ex-by) \\ \dot{y} = y(-c-fy+dx) \end{cases}$$

Let's find the fixed points:

We set $\dot{x} = \dot{y} = 0$, so

$$\left.\begin{array}{l} x(a-ex-by) = 0 \\ y(-c-fy+dx) = 0 \end{array}\right\} \Rightarrow \begin{array}{l} \underline{x=0} \text{ or } \underline{a-ex-by=0} \\ \underline{y=0} \text{ or } \underline{-c-fy+dx=0} \end{array}$$

Thus, we have four cases:

1. $x=0$ and $y=0$
2. $x=0$ and $-c-fy+dx=0$
3. $y=0$ and $a-ex-by=0$
4. $a-ex-by=0$ and $-c-fy+dx=0$

① is clearly a fixed at $\boxed{(0,0)}$

②: $x=0 \Rightarrow -c-fy=0 \Rightarrow y=-\frac{c}{f}<0$ → BUT! this is physically nonsense.... negative population.

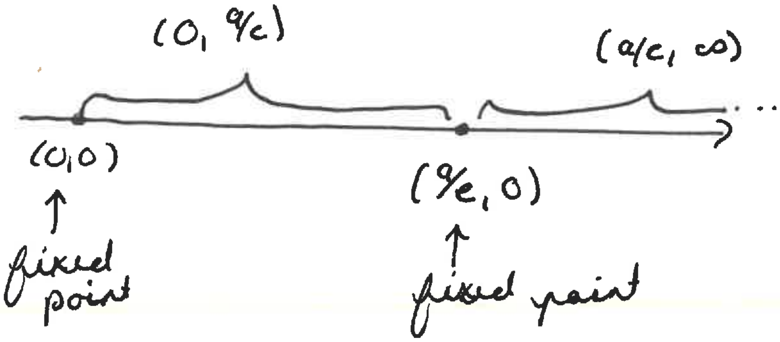
③ $y=0 \Rightarrow x=\frac{a}{e} \Rightarrow \boxed{\left(\frac{a}{e},0\right)}$

④ $$\left.\begin{array}{l} a-ex-by=0 \\ -c-fy+dx=0 \end{array}\right\} \Rightarrow \underbrace{y=\frac{ad-ce}{bd+fe},\quad x=\frac{1}{e}\left(a-\frac{b(ad-ce)}{bd+fe}\right)}.$$

Observe that these depend upon the ~~variables~~ parameters... no negatives allowed.

Thus we (for sure) have 2 fixed points: $(0,0)$ and $\left(\frac{a}{e},0\right)$: $\Big($ $(a/e,0)$ is where the growth and logistic factors cancel out $\Big)$.

Observe that, along the x-axis we have four areas/points of consideration:



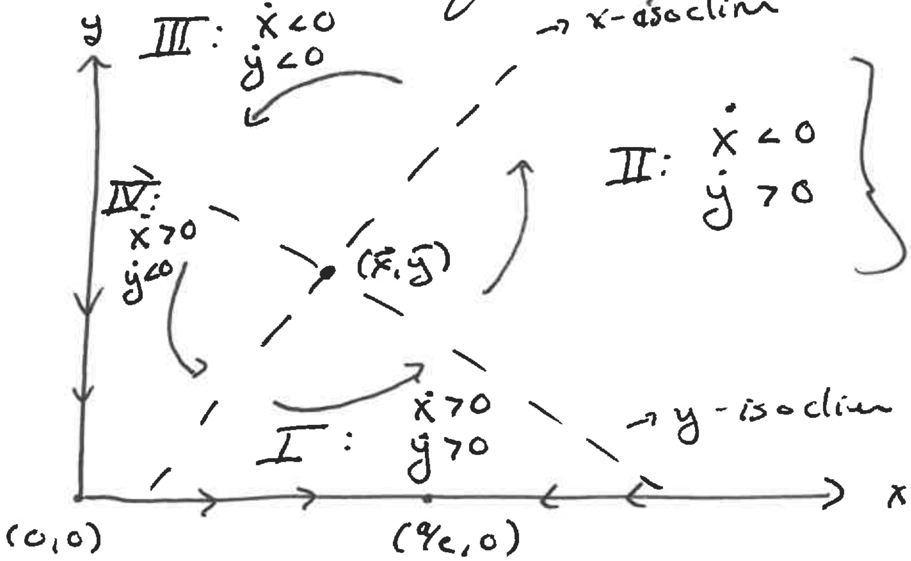
But what happens in these intervals? (ie, when there are no predators?). Naturally, we would expect logistic growth. Let's check:

when, $0<x<a/e$, $\dot{x}=x(a-ex)>0$, so $x \xrightarrow{t\to\infty} \left(\frac{a}{e},0\right)$. } verifies

when $a/e<x$, $\dot{x}=x(a-ex)<0$, so $x \xrightarrow{t\to\infty} (a/e,0)$.

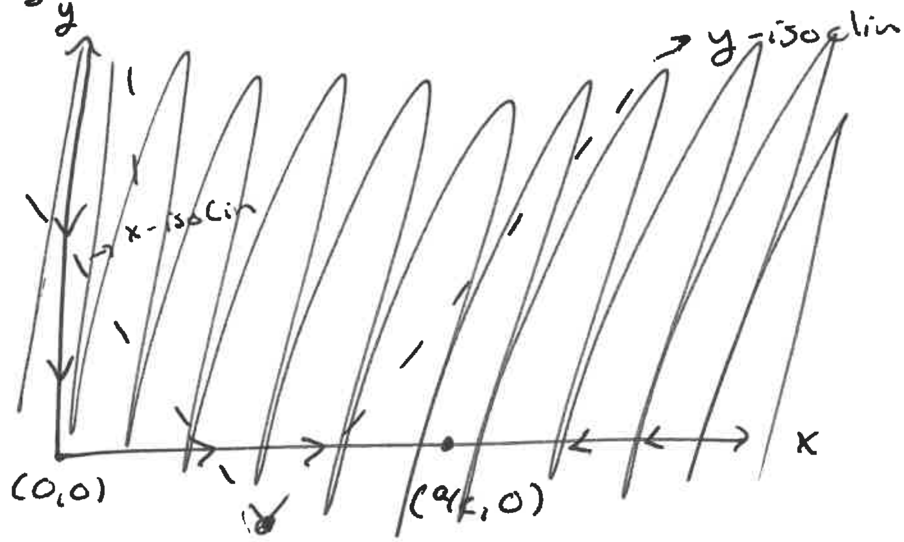
When $x=0$, what happens to y? (The predator population).

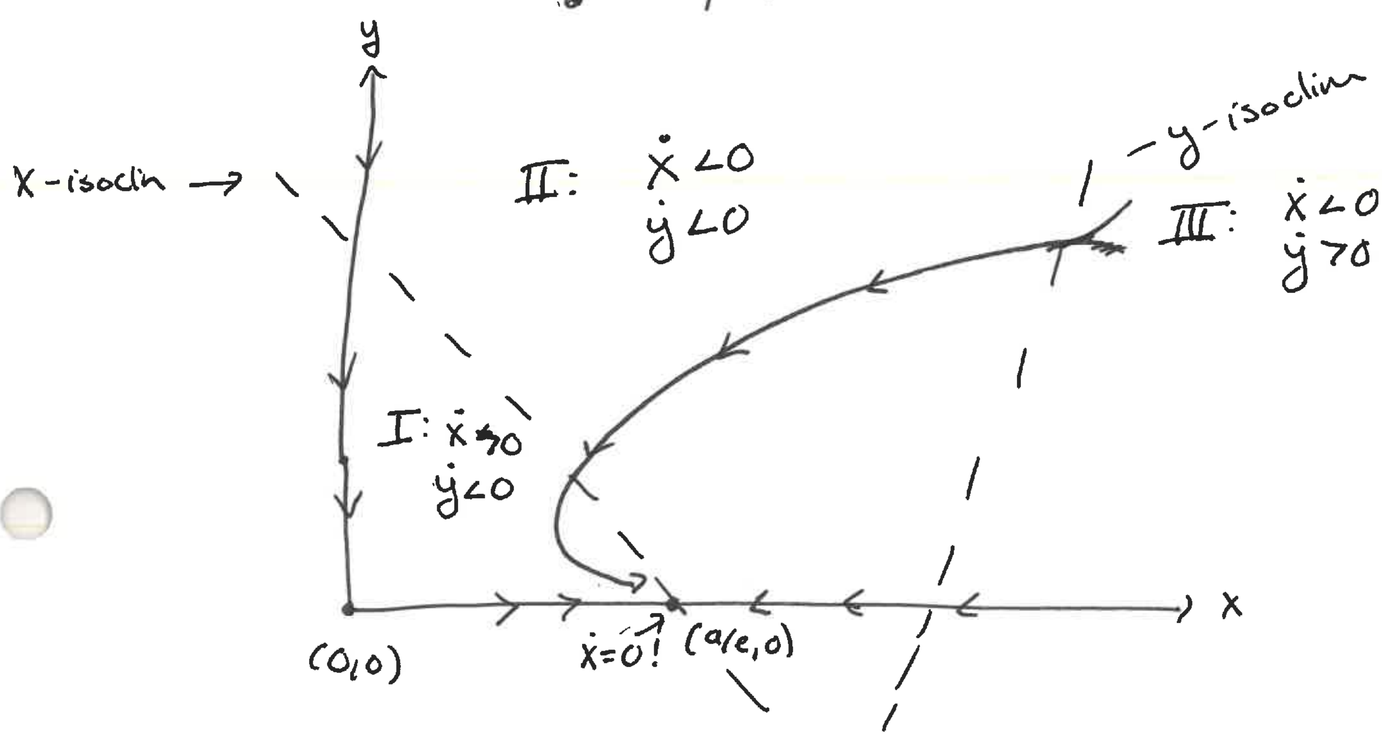
If they intersect (at the point we found earlier), in $\mathbb{R}_2^+$, we have another fixed point:



Is this periodic...?

If they don't intersect, we lack a fixed point, so:





When we do have a fixed point, it is difficult to know if the point is an attractor, a repellor, or if the solution orbits. Unfortunately, the isoclines do not settl this question.

2.6: On ω-limits and Lyapunov functions:

While a solution to an ODE (or even PDE) may exist, we may not know how to find them. So, we can either approximate (numerically), or analyze the qualitative behavior (or the dynamics). We will care about the second. The asymptotic behavior is captured by solution are found in its ω-limit.

Def Let $\dot{\vec{x}} = \vec{f}(x)$ be a time-independent ODE in $\mathbb{R}^n$; let $\vec{x}(t)$ be the solution for $t \geq 0$, w/ $\vec{x}(0) = \vec{x}_0$. The ω-limit of $\vec{x}_0$ is the set of all accumulation points of $\vec{x}(t)$ as $t \to \infty$: initial point!

$$\omega(\vec{x}_0) = \{\vec{y} \in \mathbb{R}^n : \vec{x}(t_k) \to \vec{y} \text{ for some sequen } t_k \to \infty\}.$$

So, if $\vec{x}_1 \in \omega(\vec{x}_0)$, then for each ~~[illegible]~~ $\exists$ a subsequen $\{t_{k_j}\}_1^\infty$ s.t. $\vec{x}(t_{k_j}) \to \vec{x}_1$. Equivalently, the solution will continue will continue to visit every neighborhood of $\vec{x}_1$.

Note that if $K \subseteq \mathbb{R}^n$ is compact, and ~~$\vec{x}(t)$~~ $dm(\vec{x}(t)) \subseteq K$, then $\omega(\vec{x}) \neq \phi$ (such a convergent subsequen exists).

?

~~If $\vec{z}$ is on the orbit of ~~[illegible]~~ $\vec{x}(t)$, ~~then they all~~ ~~share the same~~, then $\vec{z} \in \omega(\vec{x})$, as ~~to~~ $\vec{x}(t) = \vec{z}$ for some time~~

any point $\vec{z}_0$ on the orbit of $\vec{x}_0$ (initial data) has the same ω-limit as $\vec{x}_0$. Indeed, $\vec{z}_0 = \vec{x}(T)$ for some T. If $\vec{x}(t_k) \to y$, then so does

$$\vec{x}(t_k + T) = \vec{z}(t_k).$$

↳ ODE solution for ~~in~~ z_0

$$\vec{x}(t_k) = \vec{z}(t_k - T).$$

Clearly, $\omega(\vec{x}_0)$ is closed:

$$\omega(\vec{x}_0) = \bigcap_{t \geq 0} \overline{\{\vec{x}(s) : s \geq t\}}.$$

$\omega(\vec{x}_0)$ is also invariant:

Let $y \in \omega(\vec{x}_0)$, $t > 0$. Since $\vec{x}(t_k) \to y$ for some $\{t_k\}_1^\infty$ and solutions are continuous w/ their data, we have:

$$\vec{x}(t_k + t) \to \underbrace{\vec{y}(t)}_{\text{solution for initial } \vec{y}_0 =: \vec{y}}. \text{ Thus, } \vec{y}(t) \in \omega(\vec{x}_0) \ \forall t.$$

Rest points and periodic limits are their own ω-limits: for z_0 fixed point:

$$\omega(z_0) = \{z_0\}.$$

$$\forall t, \quad \vec{x}(t) \in \omega(\vec{x}_0).$$

If $\omega(x_0)$ is compact, it is connected.

Theorem: Let $\dot{\vec{x}} = \vec{F}(\vec{x})$ be a time-independent ODE on $G \subseteq \mathbb{R}^n$. Let $V: G \to \mathbb{R}$ be continuously differentiable. If for some solution $X(t)$, the derivative of the map $t \to V(x(t))$ ($\dot{V}$) satisfies $\dot{V} \geq 0$, then $\omega(\vec{x}_0) \cap G$ is contained in the set $\{\vec{x} \in G : \dot{V}(x) = 0\}$.

Proof: Let $y \in \omega(\dot{x}_0) \cap G$. Then, $\exists \{t_k\}_1^\infty$ s.t. $X(t_k) \to y$.

Since $\dot{V} \geq 0$ along the orbit of X_0, then $\dot{V}(y) = \lim_{k\to\infty} V(x(t_k)) \geq 0$.

Suppose $\dot{V}(y) > 0$. We claim that $V(\vec{y}(t)) > V(\vec{y}_0)$ for all t.

Since $\dot{V}$ is continuous, we have $\dot{V}(y(t)) = \lim \dot{V}(x(t_k+t)) \geq 0$.

Thus, $\dot{V}(y(t)) \geq 0$. Since $\dot{V}(y) > 0$, this gives us

$$\dot{V}(y(t)) > V(y), \quad \forall t.$$

Since $V(x(t))$ is monotonically increasing, we have

$$V(x(t)) \leq V(y) \quad \forall t \text{ as well.}$$

But then, ahan alon, $V(x(t_k+t)) > V(y)$ for k large enough, which is a contradiction. So $\dot{V}(y) = 0$. Thus,

$$\omega(x_0) \cap G \subseteq \{z \in G : \dot{V}(z) = 0\}.$$

2.7: Coexistence of predators and prey

Suppose we go back to predators and preys with logistic considerations, with fixed point $(\bar{x}, \bar{y})$. Is this point stable? Let

$$V(x,y) = d\,H(x) + b\,G(y),$$

$$H(x) = \bar{x}\log(x) - x, \quad G(y) = \bar{y}\log y - y. \text{ Thus,}$$

$$\dot{V}(x,y) = \frac{\partial V}{\partial x}\cdot \dot{x} + \frac{\partial V}{\partial y}\cdot \dot{y}.$$

$$= d\left(\frac{\bar{x}}{x} - 1\right)x(a - by - ex) + b\left(\frac{\bar{y}}{y} - 1\right)y(-c + dx - fy).$$

Since $\bar{x}, \bar{y}$ are solutions to the inclines, we replace with $a = e\bar{x} + b\bar{y}$; $c = d\bar{x} - f\bar{y}$. Thus:

$$\dot V(x,y) = d(\bar x - x)(b\bar y + e\bar x - by - ex) + b(\bar y - y)(-d\bar x + f\bar y + dx - fy)$$
$$= de(\bar x - x)^2 + bf(\bar y - y)^2 \geq 0.$$

Thus, by the theorem, every ω-limit is contained in $\{(x,y) : \dot V(x,y) = 0\}$. When $f > 0$, this implies that $\bar x = x$, $\bar y = y$ is the ω-limit.

If $f = 0$, then we have $\omega(\vec x_0) \subseteq \{(x,y) : y > 0,\ x = \bar x\}$. But the set is invariant, so since $y \to \bar y$ when $x = \bar x$, we have that $(\bar x, \bar y)$ is the limit point. Thus, every solution converges to this limit point.

Def: A strict Lyapunov function is one which $\dot V(x) > 0$ when x is not a fixed point. describes steady uphill movement in 3D.

Def: A rest point z of an ODE $\dot{\vec x} = \vec f(\vec x)$ is stable if for any neighbourhood U of z ~~then~~ there exist an open neighbourhood of W such that any orbit through W remains in U forever. (ie, $x \in W \Rightarrow x(t) \in U$ for all $t \geq 0$). It is asymptotically stable if such orbit also converge to z.

Def: The set of points x with $x(t) \to z$ is called the basin of attraction of z. If it is the whole set, then z is globally stable.

Interlude on Systems of linear differential equations.

Suppose we have a system of linear differential equations:

$$\dot{X}_1(t) = a_{11}X_1(t) + \dots + a_{1n}X_n(t)$$

$$\vdots$$

or

$$\dot{X}_n(t) = a_{n1}X_1(t) + \dots + a_{nn}X_n(t),$$

$$\dot{\vec{X}} = \underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A\vec{x}. \quad (x \in \mathbb{R}^n).$$

Then, suppose v_1 is an eigenvector of A with eigenvalue λ_1 (suppose for now that $\lambda_1 \in \mathbb{R}$). Then, v_1 is a straight line solution to the ODE:

($\dot{X}(t) = A\vec{X} = \lambda\vec{x}$, so the change is in the direction that $\vec{x}$ lies on.)

Next, suppose $\vec{v}_1$ is an eigenvector with $\lambda_1 \in \mathbb{R}$ as its eigenvector. We claim that

$Y(t) := e^{\lambda_1 t}\cdot\vec{v}_1$ is a solution.

First: $\dfrac{dY}{dt} = \dfrac{d}{dt}\begin{pmatrix} e^{\lambda t}v_1 \\ e^{\lambda_1 t}v_2 \end{pmatrix} = \begin{pmatrix} \lambda e^{\lambda t}v_1 \\ \lambda e^{\lambda t}v_2 \end{pmatrix} = \lambda\begin{pmatrix} e^{\lambda t}v_1 \\ e^{\lambda t}v_2 \end{pmatrix} = \lambda Y.$

Second:

$$AY = A e^{\lambda t}\vec{v} \underset{\substack{\uparrow \\ \text{scalar}}}{=} e^{\lambda t}A\vec{v} = e^{\lambda t}\lambda\vec{v} = \lambda Y. \Rightarrow$$

$$\frac{dY}{dt} = \lambda Y = AY.$$

Thus $Y(t) := e^{\lambda t}\vec{v}_1$ is a solution.

Note: The dynamics (how the solution travels along the lines) and the other solutions require further analysis.

~~This~~ Let's consider how the eigenvalues affect behavior of these straight line solutions.

① When $\lambda_1, \lambda_2 < 0$.
Let $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ be an eigenvector for λ.

Then, $e^{\lambda t}\vec{v}$ is a solution, but as $t \to \infty$ $e^{\lambda t} \to 0$, then $e^{\lambda t}\vec{v} \to 0$. Thus, the solution goes to the fixed point. (sink)

② When $\lambda_1, \lambda_2 > 0$.
Using the above, ~~and~~ instead of $e^{\lambda t} \to 0$, $e^{\lambda t} \to \infty$ now, so 0 is a source. (when all ~~points~~ $\lambda > 0$)

③ If we $\lambda_1 > 0$, but $\lambda_2 < 0$,
one solution goes to 0, but the other goes away. (Saddle point).

When λ is complex:

The argument above shows that $e^{\lambda t} v_1$ is a solution. But now this is not a straight line through space, but a spiral, ~~and~~ and:

1) If $Re(\lambda) < 0$, then a sink spiral
2) If $Re(\lambda) > 0$, then a source spiral
3) If $Re(\lambda) = 0$, then the solution is an orbit.

When λ is a multi-root:

(We will only note the general solution)

Consider $\frac{dY}{dt} = AY$, where A has repeated ~~root~~ eigenvalue λ. Let $\vec{v_1}$ be the eigenvector:

$$Y(t) = e^{\lambda t}\underbrace{V_0}_{\text{initial condition.}} + te^{\lambda t}\vec{V_1}$$

3: The Lotka-Volterra equation for two competing species

3.1: Linear differential equations

- While most natural differential equations are non-linear, we can analyze the stability of rest points using a reduced linear equations.

Def: Let A be a real $n \times n$ matrix and

$$\dot{\vec{X}} = A\vec{x}.$$

or

$$\begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \iff$$

$$\begin{cases} \dot{x}_1(t) = a_{11}x_1(t) + \dots + a_{1n}x_n(t) \\ \vdots \\ \dot{x}_n(t) = a_{n1}x_1(t) + \dots + a_{nn}x_n(t) \end{cases} \to \text{n-variables}$$

The solution $\vec{x}(t)$ can be written as

$$e^{At}x(0), \quad \text{where}$$

$$e^{At} = I + A\frac{t}{1!} + A^2\frac{t^2}{2!} + \dots$$

?

The components $x_i(t)$ of the solutions of (3.1) are linear combinations (so constant coefficients) of following functions:

(i) $e^{\lambda t}$, when λ is an eigenvalue of A,

(ii) $c_1 e^{at}\cos(bt) + c_2 e^{at}\sin(bt)$, that is the real and imaginary parts of $e^{\mu t}$, when $\mu = a + ib$ is a complex eigenvalue of A.

(iii) $t^j e^{\lambda t}$, $t^j e^{at}\cos(bt)$, $t^j e^{at}\sin(bt)$, with $0 \le j < m$, if the eigenvalue λ or μ occurs with multiplicity m.

Note that (clearly) $\vec{0}$ is a fixed point. We can characterize fixed points as follows:

(a) a sink, if the real parts of the eigenvalues are negative. In that case, $e^{\lambda t} \to 0$ and $e^{at} \to 0$ for $t \to \infty$. Thus $\{0\}$ is the ω-limit of every orbit;

(b) a source, if the real parts of the eigenvalues are all positive. In this case $\{0\}$ is an α-limit of the orbit.

(c) a saddle, if some eigenvalues are in the left half-plane of $\mathbb{C}$ and some in the right half-plane, but none on the imaginary axis.

Def: The rest point 0 is called hyperbolic if it is a source, saddle, or sink, i.e., if no eigenvalue has real part 0.

→ Eigenvalues lying on the imaginary axis ~~are~~ correspond to "degenerate" solutions.

→ an eigenvalue 0 corresponds to a linear manifold of rest points (i.e., like a plane in $\mathbb{R}^d$).

→ a pair of purely imaginary eigenvalues $\pm ib$ ~~is~~ corresponds to a linear manifold of periodic orbits with period $2\pi/b$.

→ If all eigenvalues are on the imaginary axis, then 0 is called a center.

→ If 0 is not hyperbolic, then an arbitrarily small perturbation of the coefficients of A can change 0 into a sink, source, or saddle, and so lead to ~~a~~ vastly different behavior.

→ The hyperbolic case is structurally stable.

3.2: Linearization

3.1 More on Linear Differential equations:

Suppose we have a system of ODEs:

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + \dots + a_{1n}x_n \\ \dot{x}_2 &= a_{21}x_1 + \dots + a_{2n}x_n \\ &\vdots \\ \dot{x}_n &= a_{n1}x_1 + \dots + a_{nn}x_n \end{aligned} \Rightarrow \dot{\vec{x}} = A\vec{x}.$$

We first consider the eigenvalues and eigenvectors of this system. Let λ be an eigenvalue corresponding to eigenvector $\vec{v}$. We claim that $e^{\lambda t}\cdot\vec{v}$ is a solution to the ODE:

$$\frac{d\, e^{\lambda t}\vec{v}}{dt} = \lambda e^{\lambda t}\vec{v};$$

$$A e^{\lambda t}\vec{v} = e^{\lambda t}A\vec{v} = e^{\lambda t}\lambda\vec{v} = \lambda e^{\lambda t}\vec{v} \Rightarrow$$

$$\frac{d\, e^{\lambda t}\vec{v}}{dt} = A\vec{v}e^{\lambda t}.$$ Thus is a solution.

Hence, if $\lambda \in \mathbb{R}$, $\lambda < 0$, then $e^{\lambda t}\vec{v} \longrightarrow \vec{0}$, and if $\lambda \in \mathbb{R}$, $\lambda > 0$, $e^{\lambda t}\vec{v} \to \infty$. If λ is complex, then we have a spiral, and in that case, if $Re(\lambda) < 0$, we spiral down to 0, and if $Re(\lambda) > 0$, we spiral away from 0.

Now, we can get a set of solutions

3.2: Linearization

Consider the system $\dot{x} = \vec{f}(x)$ a point $\vec{z} \in \mathbb{R}^n$.

→ If z is not a rest point, then exists a neighborhood U of z where the orbits can be "straightened out."
Follows from the Taylor expansion of $\vec{f}$ near z and that $\vec{f}(\vec{z}) \neq 0$.

→ If z is a rest point, a little more tricky. The next term of the Taylor Expansion is the first order Jacobian:

$$A := \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\vec{z}) & \cdots & \frac{\partial f_1}{\partial x_n}(\vec{z}) \\ \vdots & & \\ \frac{\partial f_n}{\partial x_1}(\vec{z}) & \cdots & \frac{\partial f_n}{\partial x_n}(\vec{z}) \end{bmatrix},$$

and the equation $\dot{\vec{y}} = A\vec{y}$ can easily be solved.
How does this help us understand the orbits around z? Need the following:

Theorem: For any hyperbolic rest point $\vec{z}$ of

$$\dot{\vec{x}} = \vec{f}(\vec{x})$$

there exists a neighborhood of z U, a neighborhood of 0 V, and a homeomorphism $h: U \to V$ s.t.

$$y = h(x) \Rightarrow y(t) = h(x(t)) \text{ for all } t \in \mathbb{R},\ x(t) \in U.$$

Basically, if we can understand $\dot{\vec{y}} = A\vec{y}$ at the origin, we understand $\dot{\vec{x}} = \vec{f}(\vec{x})$ at $\vec{z}$.

Discrete systems: If z is a hyperbolic fixed point of $\vec{x} \to \vec{f}(\vec{x})$, then the local behavior is completely reflected by $\vec{x} \to \underbrace{D_z}_{\text{matrix}}\vec{f}(\vec{x})$:

There exists a homeomorphism g from neighborhoods U & V s.t. for all $\vec{x}$.

$$g(\vec{x}) = D_z \vec{f}(g(\vec{x})),$$

and so by understanding $D_z\vec{f}$ near $\vec{0}$ gains understanding near $\vec{z}$.

Euler Scheme: Suppose we have $\dot{\vec{x}} = \vec{f}(\vec{x})$. The natural differ equation is, w/ step h.

$$\vec{x}_{n+1} = \vec{x}_n + h\vec{f}(\vec{x}_n)$$

↪ $\dfrac{\vec{x}_{n+1} - \vec{x}_n}{h} = \vec{f}(\vec{x}_n)$, a differ approximation.

If h is small enough, then the orbits of the differ scheme will be similar to those of the differential equation, but for finite time. Indeed, compare:

$$\underbrace{\dot{\vec{y}} = A\vec{y}}_{(0)}; \quad \underbrace{y_{n+1} = y + hAy}_{(1)}$$

$\vec{0}$ is stable for (0) iff all eigenvalues of A have negative real part, i.e., $\Re(\lambda) < 0$.

If λ is an eigenvalue of A, then $1+h\lambda$ is an eigenvalue of (1). Thus, for stability of (1) we need,

$$|1+h\lambda| < 1$$ (need to approach a value).

ie, $\lambda \in (-1/h, 1/h)$. Thus, λ has more restrictions in the difference scheme than the ODE case.

→ Stability in (0) implies stability in (1) if h is small enough.

→ If h is too large, then 0 losses the stability.

<u>Ex:</u> If $\lambda < 0$, and $h > -\frac{2}{\lambda}$, then overshooting will occur.

→ If $\vec{0}$ is a center for (0), then it will be unstable for (1), regardless of h.

<u>3.3: A competition Equation</u>

→ Two competing species

$$\dot{x} = x(a-bx-cy)$$
$$\dot{y} = y(d-ex-fy)$$

$\Rightarrow$ ~~$J(x,y) = [-b$~~

→ clinelines: $a-bx-cy=0$
$d-ex-fy=0$

→ Fixed point: $\bar{x} = \frac{af-cd}{bf-ce}$; $\bar{y} = \frac{bd-ae}{bf-ce}$.

The Jacobian at $(\bar{x}, \bar{y})$ is: Easy to check.

$$J(\bar{x},\bar{y}) = \left.\begin{bmatrix} a-2bx-cy & -cx \\ -ey & d-e-2fy \end{bmatrix}\right|_{\substack{x=\bar{x}\\ y=\bar{y}}} = \begin{bmatrix} -b\bar{x} & -c\bar{x} \\ -c\bar{y} & -f\bar{y} \end{bmatrix}$$

<u>2 Cases</u>: Suppose $(\bar{x}, \bar{y}) \in \text{int}(\mathbb{R}^2_+)$. Then,

if $bf > ce$, we have that $af - cd > 0$, ~~…~~, $bd - ae > 0$,

so $\frac{b}{e} > \frac{c}{f}$, $\frac{a}{d} > \frac{c}{f}$, $\frac{b}{c} > \frac{a}{d}$, so.

$$\frac{b}{e} > \frac{a}{d} > \frac{c}{f}.$$

We now consider when $y > \bar{y}$ (we only consider this case - the others are similar.) By the inequalities, we get

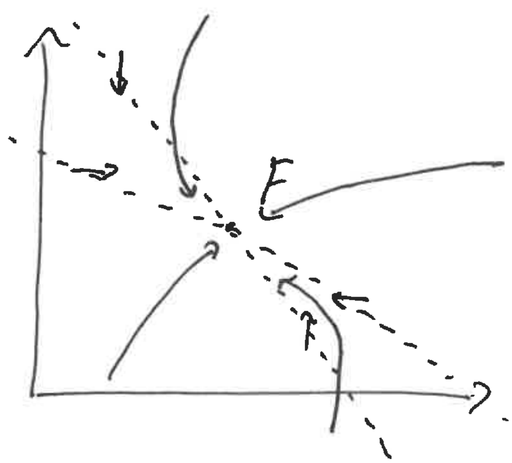
$ce - bf < 0$, so $y(ce-bf) < \bar{y}(ce-bf) \Rightarrow$

$y(ce-bf) < \frac{bd-ea}{bf-ce} \cdot (ce-fb) \Rightarrow y(ce-bf) < -(bd-ea)$. So, if

we substitute $a - bx - cy = 0$ into $d - ex - fy$, we get

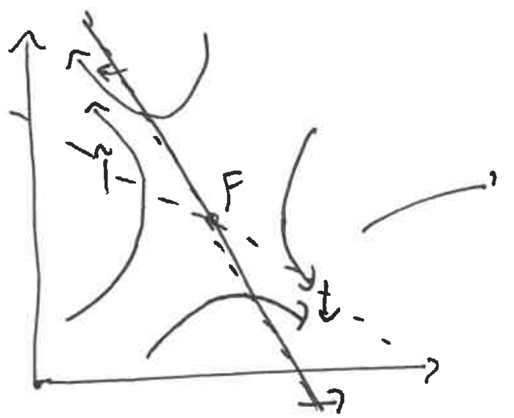
$\frac{bd - ea + ecy - fby}{b} = \frac{bd - ea + y(ce-fb)}{b} < 0$. Thus, we

get a sink (after we try all others.)



b) Otherwise:

$$\frac{c}{f} > \frac{a}{d} > \frac{b}{c}$$



3.4: Cooperative systems

<u>Def</u>: an ODE $\vec{f}(\vec{x}) = \dot{\vec{x}}$ is <u>cooperative</u> if

$$\frac{\partial f}{\partial x_j}(\vec{x}) \geq 0 \;\; \forall\, \vec{x} \in G \subseteq \mathbb{R}^n \;\; \forall i \neq j.$$

<u>Theorem</u>: The orbits of two-dimensional cooperative system converge either to a rest point or to infinity.

<u>Proof</u>: Suppose $\dot{X}(t_0) \in \mathbb{R}^+$ for some t_0, then $\dot{X}(t) \in C_1$ for all $t \geq t_0$. If (for contradiction), $\dot{X}_1(t) = 0$,

but $\dot{x}_2(t) \geq 0$; then

Chain Rule

$$\ddot{x}_1 = \dot{f}_1(x(t)) = \frac{\partial f_1}{\partial x_1}\dot{x}_1 + \frac{\partial f_1}{\partial x_2}\dot{x}_2 \geq 0,$$

so $\dot{x}(t_0)$ cannot become negative. A similar argument works for $\mathbb{R}^-$. If $\dot{x}(t_0)$ belongs to QII or QIV, then it must belong to $\mathbb{R}^+$ or $\mathbb{R}^-$ in some amount of time. Thus, $x_1(t)$, $x_2(t)$ are ultimately monotone, and so either go to infinity or converge. ∎

Chapter 4: Ecological Equations for Two species

4.1: The Poincare-Bendixson theorem

Jordan-Curve Theorem: Any periodic orbit γ in the plane divides the plane into two disjoint connected sets, the interior and exterior. Two points in the interior (or exterior) can be connected by a path which never intersects γ, but every path between the interior and exterior must cross γ.

→ The idea is ~~to~~ if we have a limit point then we must be getting close in time. But, if we don't have an ~~attractor~~ rest point, we don't have an attractor. But if we don't have an attractor, then the only way to "get close" is if we never leave, i.e., $\omega(x) = \gamma$.

Poincare-Bendixson theorem: Let $\dot{\vec{x}} = \vec{F}(\vec{x})$ be an ODE defined on an open set $G \subseteq \mathbb{R}^2$. Let $\omega(\vec{x})$ be a nonempty compact ω-limit set. Then, if $\omega(\vec{x})$ contains no rest point, it must be a periodic orbit.

→ a fixed point is like an attractor, so if there are no attractors, then the whole path must be $\omega(\vec{x})$.

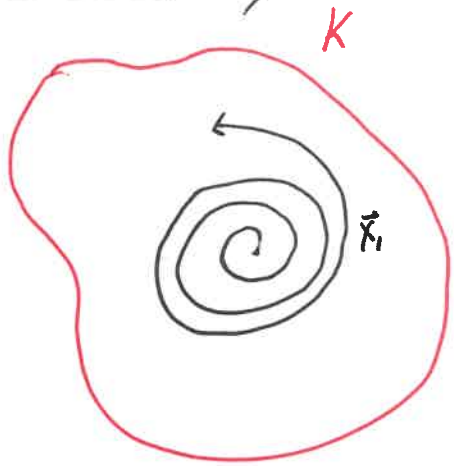
Consequence: If $K \subseteq G$ is nonempty, compact and forward invariant, then K must contain a rest point or periodic orbit. If γ is a periodic orbit which, together with its interior Γ, is contained in G, then Γ contains a rest point. $\Rightarrow$ Why???

Picture:

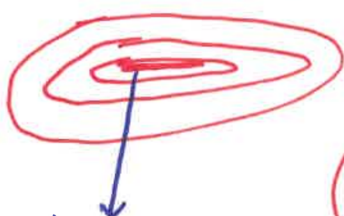
Must contain $\omega(\vec{x})$ nonempty and compact $\Rightarrow$ apply theorem!

$\Downarrow$

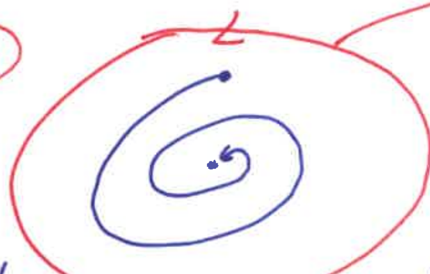
$\vec{X}_1$ = invariant orbit



$\Rightarrow$ For some $\vec{X}_0$, $\vec{X}_0$ is either a rest point, or gives rise to a periodic orbit.



"attractor"
$\Rightarrow$ rest point.



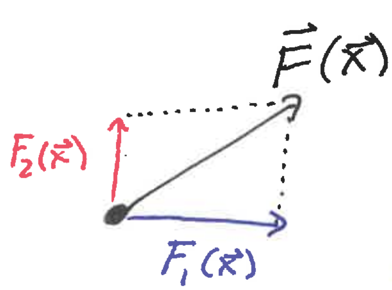
Periodic orbit γ.

$\Rightarrow$ For some $\vec{X}_0$ in the interior, $\vec{X}_0$ is either a fixed point or gives rise to another periodic orbit.

Bendixson-Dulac Method: An ODE $\dot{\vec{x}} = \vec{F}(\vec{x})$ defined on a simply connected set G such that $\operatorname{div}(\vec{F}(\vec{x})) \neq 0$ for all $\vec{x} \in G$ admits no periodic orbit. (Where

$$\operatorname{div}(\vec{F}(\vec{x})) = \frac{\partial F_1}{\partial x_1}(\vec{x}) + \frac{\partial F_2}{\partial x_2}(\vec{x}).\ )$$

Picture:



$\frac{\partial F_1(\vec{x})}{\partial x_1}$ = How fast F_1 is changing.

$\frac{\partial F_2(\vec{x})}{\partial x_2}$ = How fast F_2 is changing.

If γ is a periodic orbit with interior Γ, then Γ is an invariant set, and so its area is preserved under flow (or reduces), but neither is possible, since $\operatorname{div}(\vec{f}(\vec{x})) > 0$ implies that area is expanding.

Greens Theorem:

$$\oint_C (L\,dx + M\,dy) = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dx\,dy.$$

$$\Rightarrow \int_\Gamma \operatorname{div}(f)\, d(x_1, x_2) = \int_\Gamma \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\, d(x_1, x_2)$$

Here $x_2 = g_1(x_1)$; $x_1 = h(x_2)$ to get path integral.

$$= \int_\gamma f_2(\vec{x})\, dx_2 - f_1(\vec{x})\, dx_1$$

Chain rule.

Change variables $\rightarrow$ $= \int_0^T \left[f_2(\vec{x}(t))\dot{x}_1 - f_1(\vec{x}(t))\dot{x}_2\right] dt.$

Since $\operatorname{div}(f) > 0$, LHS > 0, but $f_1 = \dot{x}_1$, $f_2 = \dot{x}_2$ (by assumption) so RHS $= 0$.

Dulac function.

Corollary: If there exists some positive B on G s.t. the vector field $B\vec{f}$ has positive divergence at every point, then $\dot{\vec{x}} = \vec{f}(\vec{x})$ admits no periodic orbits.

4.2: Periodic Orbits for 2D Lotka-Volterra

<u>Theorem</u>: The 2D Lotka-Volterra equation:

$$\dot{x} = x(a+bx+cy)$$
$$\dot{y} = y(d+ex+fy)$$

admits no isolated periodic orbits.

<u>Proof</u>: Let γ be a periodic solution, let Γ be the interior of γ. Then, from last time Γ has a rest point. Therefore the isoclines

$$a+bx+cy=0; \rightarrow x\text{-isocline}$$
$$d+ex+fy=0; \rightarrow y\text{-isocline}$$

must intersect in Γ. Then, the rest point is given by:

$$\bar{x} = \frac{dc-fa}{bf-ce}; \quad \bar{y} = \frac{ae-bd}{bf-ce}. \Rightarrow bf-ce \neq 0.$$

We want to apply Bendikson-Dulac Theorem. Use the Dulac function:

$$B(x,y) = x^{\alpha-1}y^{\beta-1}. \rightarrow \text{Will choose } \alpha, \beta \text{ later.}$$

Then:
$$\frac{\partial B(x,y)\dot{x}}{\partial x} = \frac{\partial}{\partial x}(x^{\alpha}y^{\beta-1}(a+bx+cy))$$
$$= \alpha x^{\alpha-1}y^{\beta-1}(a+bx+cy) + b x^{\alpha}y^{\beta-1}.$$

$$\cancel{\frac{\partial B(x,y)\dot{y}}{\partial y} = \frac{\partial}{\partial y}(x^{\alpha}y^{\beta-1}(d+ex+fy))}$$
$$\cancel{= (\beta-1)y^{\beta-2}x^{\alpha}(d+ex+fy) + f x^{\alpha}y^{\beta-1}.}$$

$$\cancel{\text{div}(BF) =}$$

$$\frac{\partial B(x,y)\cdot \dot{y}}{\partial y} = \frac{\partial}{\partial y}\left(x^{\alpha-1}y^{\beta}(d+ex+fy)\right)$$

$$= \beta x^{\alpha-1}y^{\beta-1}(d+ex+fy) + f x^{\alpha-1}y^{\beta}$$

$$\Rightarrow \operatorname{div}(Bf) = \alpha x^{\alpha-1}y^{\beta-1}(a+bx+cy) + b x^{\alpha}y^{\beta-1} + \beta x^{\alpha-1}y^{\beta-1}(d+ex+fy) + f x^{\alpha-1}y^{\beta}.$$

$$= B[\alpha(a+bx+cy) + bx + \beta(d+ex+fy) + fy].$$

Now consider the matrix: $\begin{bmatrix} b & e \\ c & f \end{bmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$. By the above, it has nonzero determinant, and thus is a bijection.

Thus, we may choose α, β such that

$$\begin{bmatrix} b & e \\ c & f \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -b \\ -f \end{bmatrix} \iff \left.\begin{matrix} \alpha b + \beta e = -b \\ \alpha c + \beta f = -f \end{matrix}\right\}.$$

Then, $\alpha(a+bx+cy) + bx + \beta(d+ex+fy) + fy =$

$\alpha a + \alpha b x + \alpha c y + bx + \beta d + \beta e x + \beta f y + fy =$

$\alpha a + (\alpha b + \beta e)x + (\alpha c + \beta f)y + bx + \beta d + fy =$

$\alpha a - bx - fy + bx + \beta d + fy =$

$\alpha a + \beta d.$

$$\Rightarrow \operatorname{div}(Bf) = B(\alpha a + \beta d) := B\delta$$

Since we assumed a periodic orbit, we have that the divergence must be 0. Thus:

$$\operatorname{div}(Bf) = B\delta = 0 \quad \forall (x,y) \in \mathbb{R}^2 \Rightarrow \boxed{\delta = 0}.$$

Thus we have

$$-\frac{\partial}{\partial x}(B\dot{y}) = -\frac{\partial}{\partial y}(B\dot{x}).$$

This is the 2D integrability condition for the vector field $(B\dot{x}, -B\dot{y})$. → Not sure what this means; differential topology

Thus, there exist some V on Γ such that

$$\frac{\partial V}{\partial x} = B\dot{y}, \quad \frac{\partial V}{\partial y} = -B\dot{x}$$

→ Two part magic trick:
a.) Dulac function
b.) Dulac gives existence of invariant of motion.
↳ Don't need to pull one out of our hat.

Thus:

$$\dot{V} = \frac{\partial V}{\partial x}\cdot\dot{x} + \frac{\partial V}{\partial y}\cdot\dot{y} = B\dot{y}\dot{x} - B\dot{y}\dot{x} = 0.$$

Thus, V is an invariant of motion in Γ. Thus, if a periodic exists, it is surrounded by other periodic orbits.

↳ I find this wanting: how do I know that this new orbit is close to the original?

4.3: Limit cycles and the Gause Predator Prey model:

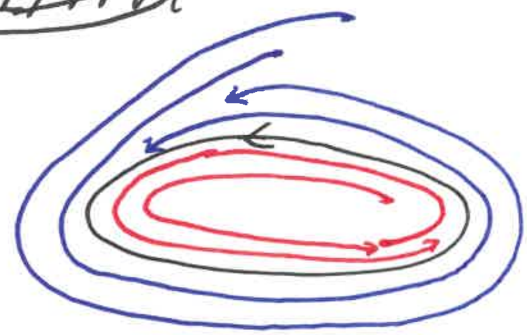
Def: a periodic orbit γ is an attractor if $\omega(x) = \gamma$ for all initial conditions x in some neighborhood of γ; it is a limit cycle if $\omega(x) = \gamma$ for at least $x \notin \gamma$.

→ Does not occur for linear systems.

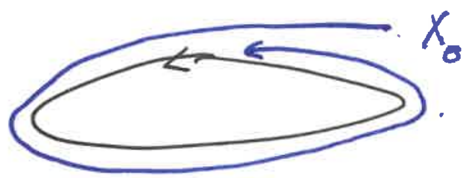
→ as just shown, does not hold for Lotka-Volterra (periodic orbits are not isolated).

Picture:

Attractor:



Limit Cycle:



→ Look at nonlinear systems.

→ Let x, y be densities;

→ with no predators, prey should converge towards a limit, $K > 0$. • Thus, $\dot{x} = x g(x)$, when 1) $g(x) > 0$ when $x < K$ → increase

2) $g(x) < 0$ when $x > K$ → decrease

3) $g(K) = 0$ → Fixed point

• Predators will reduce rate: 1) $y p(x)$, p = # killed per predator

2) $p(0) = 0$

3) $p(x) > 0$ $x > 0$ ↗ more prey.

• Predator system given by: ~~d+p~~(x). $-d + q(x)$;

$-d$ = mortality rate

$q(x) > 0$, increasing.

• $q(0)=0$,

• $\frac{d}{dx} q(x) > 0$.

⟹ System:

$$\left.\begin{aligned} \dot{x} &= xg(x) - yp(x) \\ \dot{y} &= y(-d + q(x)) \end{aligned}\right\}$$ g, p, q, continuous...?

→ again $x \equiv 0$, $y \equiv 0$ are solutions. ⟹ $\mathbb{R}^+_2$ is invariant.

→ Since $q(x)$ is increasing, $\dot{y}=0$, $y \neq 0$ ⟹ $-d+q(x)=0$ for only 1 x; call it $\bar{x}$. If such an $\bar{x}$ does not exist, then $-d+q(x)<0$, and so $\dot{y}<0$, and predators will die. Thus, we assume $\bar{x}$ exists.

→ The x-isocline is given by $y = x\frac{g(x)}{p(x)}$. Since $g(x)<0$ when $x>K$, we have at most 1 intersection. If no intersection, then ($\bar{x} \geq K$), predators will vanish ($q(x)<d$ when $x<\bar{x}$). If the intersection does occur ($\bar{x}<K$), then we have a unique rest point.

→ On the boundary 2 unique rest points: $(0,0)$, $(K,0)$.

→ $\mathbb{R}^+$ is stable for $P :=: (K,0)$ ↗ What? How?

→ Unstable manifold is an orbit.

→ Consider x on the orbit: $\omega(x)$ is nonempty & compact.

⟹ Poincaré Bendixson implies:

a.) If $\omega(x)$ has no rest point, then it has a periodic orbit; $(\bar{x}, \bar{y})$ must remain inside the orbit.

b.) If there is a rest point, it must be $(\bar{x}, \bar{y})$ in fact globally stable.